

ABSTRACT

We present the discrete circular convolution and its algorithms of calculus. Using discrete Fourier transform can be determined the circular deconvolution, ie the circular convolution inverse. We exemplify these concepts by solving an equation of circular convolution. This type of convolution product is applies to the periodic phenomena, discretely analyzed. By this paper we continue the series of articles about the types of discrete convolution products and their applications, some of which being cited in the References.

Keywords: Circular convolution, circular deconvolution, discrete Fourier transform, circular convolution equations.

1. DISCRETE CIRCULAR CONVOLUTION

Being given two finite sequences $x = (x_0, x_1, \dots, x_{N-1})$ and $y = (y_0, y_1, \dots, y_{N-1})$ of the same of the same length N , is called *discrete circular convolution of period N* the sequence $z = x \otimes y = (z_0, z_1, \dots, z_{N-1})$, where

$$z_n = \sum_{k=0}^{N-1} x_k y_{n-k \pmod{N}} = \sum_{k=0}^n x_k y_{n-k} + \sum_{k=n+1}^{N-1} x_k y_{n-k+N}, \quad n = 0, 1, \dots, N-1. \quad (1)$$

We make the convention $\sum_{k=n}^m x_k = 0$, if $m < n$.

The circular convolution can be considered also for sequences of different lengths, by right completion with 0, such that the two sequences to be of the same length.

2. ALGORITHMS FOR CALCULUS OF DISCRETE CIRCULAR CONVOLUTION

We give three types of circular convolution algorithms, which we will exemplify by the example $(1, 2, 4, 5, 6) \otimes (7, 3, 9, 8) = (102, 111, 91, 73, 109)$.

Circular algorithm

¹ Faculty of Applied Sciences, Polytechnic University of Bucharest, cirnumircea@yahoo.com

The elements of the longer sequence sits outside on a circle in direct sense, i.e. contrary to clockwise motion and the elements of the shortest in the interior of the circle, in indirect sense, i.e. in the sense of clockwise. Its first elements must stand beside one another. Multiply the elements of the two sequences that stand beside one another and add the results. Then move one position the interior sequence in the sense of clockwise and do the same operations. It is finished when the first element of the interior sequence is beside the last element of the exterior sequence. The numbers resulted by this procedure are the elements of the circular convolution.

Bilinear algorithm

The longer sequence is placed from left to right twice. Below it is placed the second sequence from right to left, its first element being under the first element of the above second sequence. Multiply the elements of the two rows that stand beside one another and add the results. Then move one position to the right the bottom sequence and do the same operations. It is finished when the first element of the bottom sequence is beside the last element of the above second sequence. The numbers resulted by this procedure are the elements of the circular convolution.

Example.

1	2	4	5	6	1	2	4	5	6	1	2	4	5	6	1	2	4	5	6
		8	9	3	7			8	9	3	7			8	9	3	7		
$32 + 45 + 18 + 7 = 102$										$40 + 54 + 3 + 14 = 111$									

1 2 4 5 6 1 2 4 5 6 8 9 3 7 48 + 9 + 6 + 28 = 91	1 2 4 5 6 1 2 4 5 6 8 9 3 7 8 + 18 + 12 + 35 = 73
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$$\begin{array}{ccccccccc} 1 & 2 & 4 & 5 & 6 & 1 & 2 & 4 & 5 & 6 \\ & & & & & & 8 & 9 & 3 & 7 \\ & & & & & & 16 & +36 & +15 & +42 = 109 \end{array}$$

Aliasing algorithm

This algorithm, the simplex, consists in performing linear complete convolution (see [1] or [6]) of the two sequences and adding to the part from resulting product with the same length as the longest sequence factor, the remaining part.

1	2	4	5	6			
7	3	9	8				

7	14	28	35	42			
	3	6	12	15	18		
		9	18	36	45	54	
			8	16	32	40	48

7	17	43	73	109	95	94	48
95	94	48					

102	111	91	73	109			

3. DISCRETE FOURIER TRANSFORM

The *Fourier transform* of the finite sequence $x = (x_0, x_1, \dots, x_{N-1})$ is the sequence $\hat{x} = (\hat{x}_0, \hat{x}_1, \dots, \hat{x}_{N-1}) = F(x)$, with

$$\hat{x}_n = \sum_{k=0}^{N-1} x_k \overline{\omega}_N^{kn}, \quad n = 0, 1, \dots, N-1, \quad (2)$$

where $\omega_N = e^{2\pi i/N}$ is the N order complex root of the unity, i.e.

$$\omega_N^N = e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1$$

and $\overline{\omega}_N = e^{-2\pi i/N} = \omega_N^{-1}$ is the complex conjugate.

Theorem 1. The inverse transform is given by the relation

$$x_j = \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}_k \omega^{jk}, \quad j = 0, 1, \dots, N-1. \quad (3)$$

Proof. Using the formula for sum of a geometric progression, we have

$$\sum_{n=0}^{N-1} \hat{x}_n \omega^{nk} = \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} x_j \overline{\omega}^{jn} \omega^{nk} = \sum_{j=0}^{N-1} x_j \sum_{n=0}^{N-1} \omega^{(k-j)n} = Nx_k + \sum_{j=0, j \neq k}^{N-1} \sum_{n=0}^{N-1} \omega^{(k-j)n} =$$

$$= Nx_k + \sum_{j=0, j \neq k}^{N-1} \frac{\omega^{(k-j)N} - 1}{\omega^{k-j} - 1} = Nx_k ,$$

because, for $j \neq k$, we have $\omega^{(k-j)N} = 1$ and $\omega^{k-j} \neq 1$. It results the formula (3).

Remark. The direct and inverse discrete Fourier transforms are matrix written $F(\hat{X}) = F_N X$, respective $F^{-1}(X) = F_N^{-1} \hat{X}$, where

$$X = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix}, \hat{X} = \begin{pmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \vdots \\ \hat{x}_{N-1} \end{pmatrix}, F_N = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \overline{\omega}_N & \overline{\omega}_N^2 & \cdots & \overline{\omega}_N^{N-1} \\ 1 & \overline{\omega}_N^2 & \overline{\omega}_N^3 & \cdots & \overline{\omega}_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{\omega}_N^{N-1} & \overline{\omega}_N^{2(N-1)} & \cdots & \overline{\omega}_N^{(N-1)^2} \end{pmatrix}$$

and

$$F_N^{-1} = \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^3 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)^2} \end{pmatrix}.$$

Particularly, for $N = 4$, we have

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \text{ si } F_4^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

4. THE DISCRETE FOURIER TRANSFORM OF THE CIRCULAR CONVOLUTION

Theorem 2. For two column matrices $X = (x_0 \ x_1 \cdots x_{N-1})^T$ and $Y = (y_0 \ y_1 \cdots y_{N-1})^T$, we have the formula of discrete Fourier transform of the circular convolution,

$$F(X \otimes Y) = \hat{X}\hat{Y} = (\hat{x}_0\hat{y}_0 \ \hat{x}_1\hat{y}_1 \cdots \hat{x}_{N-1}\hat{y}_{N-1})^T,$$

the product of the column matrices $\hat{X} = (\hat{x}_0 \ \hat{x}_1 \cdots \hat{x}_{N-1})$ and $\hat{Y} = (\hat{y}_0 \ \hat{y}_1 \cdots \hat{y}_{N-1})$ from the right hand of the above formula being performed on elements.

Proof. In conformity with above definitions and iterating the sums, an arbitrary component of the discrete Fourier transform of the circular convolution of the matrices X and Y is given by the formula

$$\begin{aligned} (F(X \otimes Y))_n &= \sum_{k=0}^{N-1} (X \otimes Y)_k \bar{\omega}^{kn} = \sum_{k=0}^{N-1} \left(\sum_{j=0}^k x_j y_{k-j} + \sum_{j=k+1}^{N-1} x_j y_{k-j+N} \right) \bar{\omega}^{kn} = \\ &= \sum_{k=0}^{N-1} \sum_{j=0}^k x_j \bar{\omega}^{jn} y_{k-j} \bar{\omega}^{(k-j)n} + \sum_{k=0}^{N-2} \sum_{j=k+1}^{N-1} x_j \bar{\omega}^{jn} y_{k-j+N} \bar{\omega}^{(k-j+N)n} = \\ &= \sum_{j=0}^{N-1} x_j \bar{\omega}^{jn} \sum_{k=j}^{N-1} y_{k-j} \bar{\omega}^{(k-j)n} + \sum_{j=1}^{N-1} x_j \bar{\omega}^{jn} \sum_{k=0}^{j-1} y_{k-j+N} \bar{\omega}^{(k-j+N)n} . \end{aligned}$$

Making the changes of variables $m = k - j$ in the first term and $m = k - j + N$ in the second term from the last formula, we have

$$\begin{aligned} (F(X \otimes Y))_n &= \sum_{j=0}^{N-1} x_j \bar{\omega}^{jn} \sum_{m=0}^{N-1-j} y_m \bar{\omega}^{mn} + \sum_{j=1}^{N-1} x_j \bar{\omega}^{jn} \sum_{m=N-j}^{N-1} y_m \bar{\omega}^{mn} = \\ &= \sum_{j=0}^{N-1} x_j \bar{\omega}^{jn} \sum_{m=0}^{N-1} y_m \bar{\omega}^{mn} = (F(X))_n (F(Y))_n . \end{aligned}$$

5. CIRCULAR CONVOLUTION EQUATIONS.

Using the discrete Fourier transform method, we solve as example a circular convolution equation:

Example. Solve the circular convolution equation of period four,

$$A \otimes X = B,$$

where $A = (1 \ 9 \ 9 \ 1)^T$ and $B = (12 + \lambda \ 12 - \lambda \ 8 + \lambda \ 8 - \lambda)^T$. To make discussion after the real parameter λ and checking of obtained solution.

Solution. We denote $B = B_1 + \lambda B_2$, where $B_1 = (12 \ 12 \ 8 \ 8)^T$,
 $B_2 = (1 \ -1 \ 1 \ -1)^T$, $X = (x_0 \ x_1 \ x_2 \ x_3)^T$ and

$F(X) = \hat{X} = (\hat{x}_0 \quad \hat{x}_1 \quad \hat{x}_2 \quad \hat{x}_3)^T$. Applying the discrete Fourier transform on the equation, this take the form

$$F(A)\hat{X} = F(B_1) + \lambda F(B_2).$$

We have

$$F(A) = F_4 A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 9 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ -8-8i \\ 0 \\ -8+8i \end{pmatrix},$$

$$F(B_1) = F_4 B_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 12 \\ 12 \\ 8 \\ 8 \end{pmatrix} = \begin{pmatrix} 40 \\ 4-4i \\ 0 \\ 4+4i \end{pmatrix},$$

$$F(B_2) = F_4 B_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 0 \end{pmatrix},$$

therefore the equation becomes

$$\begin{pmatrix} 20 \\ -8-8i \\ 0 \\ -8+8i \end{pmatrix} \begin{pmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} 20\hat{x}_0 \\ -8(1+i)\hat{x}_1 \\ 0 \\ -8(1-i)\hat{x}_3 \end{pmatrix} = \begin{pmatrix} 40 \\ 4(1-i) \\ 0 \\ 4(1+i) \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 40 \\ 4(1-i) \\ 4\lambda \\ 4(1+i) \end{pmatrix}.$$

Equating the components of the column matrices by above equation, it results the system of equations

$$20\hat{x}_0 = 40, \quad -8(1+i)\hat{x}_1 = 4(1-i), \quad 0 = 4\lambda, \quad -8(1-i)\hat{x}_3 = 4(1+i),$$

from which it is obtained the solutions

$$\hat{x}_0 = 2, \hat{x}_1 = -\frac{1-i}{2(1+i)} = \frac{i}{2}, \hat{x}_3 = -\frac{1+i}{2(1-i)} = -\frac{i}{2}$$

and the compatibility condition $\lambda = 0$. The unknown $\hat{x}_2 = p$ can not be determined, leaving a real parameter. For $\lambda \neq 0$ the convolution equation have no solution. For $\lambda = 0$, the equation has an infinity of solutions, given by the formula

$$X = F^{-1} \hat{X} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} \frac{2}{i} \\ \frac{i}{2} \\ p \\ -\frac{i}{2} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2+p \\ 1-p \\ 2+p \\ 3-p \end{pmatrix}, \forall p \in \mathbf{R}.$$

Verification

We verify the above result by circular convolution, computed by aliasing method.

$\frac{2+p}{4}$	$\frac{1-p}{4}$	$\frac{2+p}{4}$	$\frac{3-p}{4}$				
1	9	9	1				

$\frac{2+p}{4}$	$\frac{1-p}{4}$	$\frac{2+p}{4}$	$\frac{3-p}{4}$				
	$\frac{18+9p}{4}$	$\frac{9-9p}{4}$	$\frac{18+9p}{4}$	$\frac{27-9p}{4}$			
		$\frac{18+9p}{4}$	$\frac{9-9p}{4}$	$\frac{18+9p}{4}$	$\frac{27-9p}{4}$		
			$\frac{2+p}{4}$	$\frac{1-p}{4}$	$\frac{2+p}{4}$	$\frac{3-p}{4}$	

$\frac{2+p}{4}$	$\frac{19+8p}{4}$	$\frac{29+p}{4}$	8	$\frac{46-p}{4}$	$\frac{29-8p}{4}$	$\frac{3-p}{4}$	
$\frac{46-p}{4}$	$\frac{29-8p}{4}$	$\frac{3-p}{4}$					

12	12	8	8				

Remark. The above example is the particular case for $m = 8, n = 10$ of the same equation with the dates

$$A = \begin{pmatrix} 1 & m+1 & m+1 & 1 \end{pmatrix}^T \text{ and } B = \begin{pmatrix} n+2+\lambda & n+2-\lambda & n-2+\lambda & n-2-\lambda \end{pmatrix}^T.$$

The author uses such exercises with parameters for student works, especially to masters. Every student has a particular pair of parameters m and n , for example they can be the row and column on which found the place of the student in the seminar room. Thus, every student has quantitatively his individual problem to solve, but all these problems are qualitatively the same.

6. REFERENCES

- [1] M. I. Cîrnu, *Algebraic equations. New solutions for old problems*, Lambert Academic Publishing, Germany, 2013.
- [2] M. I. Cîrnu, *Calculation of convolution products of piecewise defined functions and some applications*, Journal of Information Systems and Operations Management, 6(2012) no. 1, 41-52.
- [3] M. I. Cîrnu, *A certain integral-recurrence equation with discrete-continuous auto-convolution*, Archivum Mathematicum (Brno), 47 (2011), 267-272.
- [4] M. I. Cîrnu, *Determinantal formulas for sum of generalized arithmetic-geometric series*, Boletín de la Asociación Matemática Venezolana, 43 (2011) no. 1, 25-38
- [5] M. I. Cîrnu, *Initial-value problems for first-order differential recurrence equations with auto-convolution*, Electronic Journal of Differential Equations, 2011 (2011) no. 2, 1-13.
- [6] M. I. Cîrnu, *Linear discrete convolution and its inverse*, Journal of Information Systems and Operations Management, Part I. Convolution, 4 (2010) no.1, 129-137, Part II. Deconvolution, 4 (2010) no. 2, 123-137.
- [7] M. I. Cîrnu, *First order differential recurrence equations with discrete auto-convolution*, International Journal of Mathematics and Computation, 4, 2009, 124-128.
- [8] A. V. Oppenheim, R. W. Schaffer, J. R. Buck, *Discrete-time signal processing*, Prentice Hall, 1999.