

A COMPARISON OF SOME NEW METHODS FOR SOLVING ALGEBRAIC EQUATIONS

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ABSTRACT

In this paper we compare different methods of solving algebraic equations based on resolvent polynomial equations. These methods were considered by several authors, including the second author of this article. Due to the fact that no demonstration of convergence was made for these methods and therefore the speed of convergence was not determined, this study offers some important data in terms of effectiveness.

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1. INTRODUCTION

The Newton-Raphson iteration scheme for solving algebraic equations is derived by the first-order Taylor expansion and gives a recurrence formula for the approximate solution of an equation. However, more accurate methods were developed recently, based on superior-order Taylor expansion, that reduce algebraic equations to polynomial ones, named resolvent equations. The variation of the iterations can be determined from these polynomial equations, i.e. the number that must be added to the known iteration to obtain the following one, where the chosen iteration is the root of the resolvent equation with the least absolute value. Such a method was first given by J.H.He, [5], who obtained a quadratic resolvent equation deduced by the second-order Taylor expansion. Depending on the initial data, this method can be inapplicable if the resolvent equation has complex roots. However, this limitation was overcome by D.Wei, J.Wu and M.Mei, [6], who considered a cubic resolvent equation, therefore making the restriction to the choice of initial data unnecessary.

In paper [1], see also [3], several types of resolvent equations were given. The simplistic one uses the Taylor polynomial related to the function defining the equation, considered at the previous iteration. This method, which is a direct generalization of the Newton-Raphson method, proves to be the most effective. In the same paper, were obtained, not only the methods presented by He and Wei-Wu-Mei, but also some improved versions of these. Furthermore, due to the fact that the methods for solving algebraic equations presented in this article are reduced to solving polynomial ones, it is useful to mention that new, simple methods for solving polynomial equations have been given in paper [2]. See also [3]. In the following Section, a summary of the methods for solving algebraic equations by their reduction to the polynomial form is presented. In the last part, all the aforementioned

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methods will be validated in the MATLAB environment, and the results will be presented at the end of this paper.

2. PRELIMINARY NOTES

We consider the algebraic equation

$$f(x) = 0 \quad (1)$$

where $f(x)$ is a function of real variable with derivatives up to necessary order.

Iterative methods consist of determining a sequence of iterations x_n , $n = 0, 1, 2, \dots$, that approximate the exact real root x of the equation, meaning that $\lim_{n \rightarrow \infty} x_n = x$, starting from the initial value x_0 . The most common method is the Newton-Raphson method, obtained from the first-order Taylor expansion of the function $f(x)$. The iterations of this method are determined from the relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 1, 2, \dots \quad (2)$$

using the initial data x_1 and considering $x_0 = 0$. Recent methods, specified in the References Section, determine variations of iterations x_n denoted

$$t_n = x_{n+1} - x_n, n = 0, 1, 2, \dots \quad (3)$$

starting with the initial data x_0 and x_1 , hence with the initial variation $t_0 = x_1 - x_0$.

We now consider the m -order Taylor expansion, where $m = 1, 2, \dots$ of the function f , at the point x_{n+1} , in relation to the values of the function and its derivatives at the previous iteration x_n , given by the following formula

$$f(x_{n+1}) = P_{m,n}(t_n) + R_{m,n}, n = 0, 1, 2, \dots \quad (4)$$

where $R_{m,n}$ is the remainder of the expansion, and

$$P_{m,n}(t_n) = \sum_{k=0}^m \frac{f^{(k)}(x_n)}{k!} t_n^k, n = 0, 1, 2, \dots \quad (5)$$

is the m -order Taylor polynomial in x_n with the unknown t_n .

As shown in paper [1], if it is assumed that $f(x_{n+1}) = 0, R_{m,n} = 0$, the expansion (4) takes the form

$$P_{m,n}(t_n) = \sum_{k=0}^m \frac{f^{(k)}(x_n)}{k!} t_n^k = 0, n = 1, 2, \dots \quad (6)$$

This is the simplest form of a resolvent equation, and the method which is based on this formula is referred to as the Generalized Newton-Raphson method, since for $m = 1$ in (6) we obtain (2). For $m = 2$, the resolvent equation of this method is given by

$$\frac{1}{2} f''(x_n) t_n^2 + f'(x_n) t_n + f(x_n) = 0, n = 1, 2, \dots \quad (7)$$

and for $m = 3$ we have the formula

$$\frac{1}{6} f'''(x_n) t_n^3 + \frac{1}{2} f''(x_n) t_n^2 + f'(x_n) t_n + f(x_n) = 0, n = 1, 2, \dots \quad (8)$$

If it is assumed that $f(x_{n+1}) = 0, R_{m,n} = R_{m,n-1}$, the following resolvent equation is obtained

$$P_{m,n}(t_n) + f(x_n) - P_{m,n-1}(t_{n-1}) = \sum_{k=0}^m \frac{f^{(k)}(x_n)}{k!} t_n^k + f(x_n) - \sum_{k=0}^m \frac{f^{(k)}(x_{n-1})}{k!} t_{n-1}^k = 0, n = 1, 2, \dots \quad (9)$$

The method based on this equation will be referred to as Generalized He-We-Wu-Mei method, since for $m = 2$ we have the quadratic resolvent equation introduced by J.H.He, [4], [5], and for $m = 3$ we obtain the one presented by D. Wei, J. Wu and M. Mei, [6].

In paper [1], these methods were improved by simplifying the resolvent equation (9), hence obtaining the method denoted Improved Generalized He-We-Wu-Mei method

$$P_{m,n}(t_n) + \sum_{j=1}^n f(x_j) - P_{m,0}(t_0) = \sum_{k=0}^m \frac{f^{(k)}(x_n)}{k!} t_n^k + \sum_{j=1}^n f(x_j) - \sum_{k=0}^m \frac{f^{(k)}(x_0)}{k!} t_0^k = 0, n = 1, 2, \dots \quad (10)$$

Particularly, considering $x_0 = x_1$, therefore $t_0 = 0$, we obtain the simplified resolvent equations

$$\sum_{k=0}^m \frac{f^{(k)}(x_1)}{k!} t_1^k = 0, n = 1 \quad (11)$$

and

$$\sum_{k=0}^m \frac{f^{(k)}(x_n)}{k!} t_n^k + \sum_{j=2}^n f(x_j) = 0, n \geq 2 \quad (12)$$

Notice that both in the case of the resolvent equations (6) and (11) – (12), only a single initial data is necessary in order to start the method, whereas the methods based on the resolvent equations (9) and (10) require two such data.

3. NUMERICAL RESULTS

The purpose of this study is to find the numerical solution of the problem $f(x) = 0$, with the initial values x_0 and x_1 , using the abovementioned methods and to compare these methods from the point of view of both speed of convergence and the simplicity of the formula. In order to do that, a function was created using the MATLAB environment. The function offers the numerical solution of all or just some of the methods presented, depending on the calling sequence and has the following arguments: the function $f(x)$, the initial points x_0, x_1 , the number of iterations, a vector with components from 1 to 9, representing the required methods, the precision of the result.

In the following we present the results obtained using the aforesaid Matlab function for solving two particular algebraic equations, for different initial data.

Example

We consider the equation used in [5], [6] and [1], i.e. $f(x) = x^3 - \exp(-x)$, with the unique solution $x = 0.77288295914921012475 \dots$

Given the initial values $x_0 = 0$ and $x_1 = 0.5$, with 5 iterations and an error within 10^{-10} , we obtain

Iteration Method	x_2	x_3	x_4	x_5	x_6
Newton Raphson	.854972190 5	.778710528 2	.772914269 1	.772882960 1	.772882959 1
(6), $m = 2 \equiv (7)$	.783878259 7	.772882324 2	.772882959 1	.772882959 1	.772882959 1
(6), $m = 3 \equiv (8)$	.772824071 8	.772882959 1	.772882959 1	.772882959 1	.772882959 1
J.H.He	.710222585 9	.768441369 9	.772788363 9	.772882919 9	.772882959 1
Wei-Wu-Mei	.773871200 0	.772942737 2	.772882959 1	.772882959 1	.772882959 1
(10), $m = 2$	.710222585 9	.768441369 9	.772788363 9	.772882919 9	.772882959 1
(10), $m = 3$	.773871200 0	.772942737 2	.772882959 1	.772882959 1	.772882959 1

The results show that the fastest convergent method is given by formula (8), as it gets to the solution after only 2 steps. Moreover, both the Wei-Wu-Mei and its improved version, presented by the second author of this article, obtain the numerical solution with a precision of 10 digits after only 3 steps. Nevertheless, the improved version, (10), has an advantage regarding the simplicity of the formula, as it does not require the use of derivatives in previous iterations, but only in the first one, x_0 . Now, in order to compare all the methods, we have to consider $x_0 = x_1 = 0.5$ as the simplified method requires it, obtaining

Iteration Method	x_2	x_3	x_4	x_5	x_6
Newton Raphson	.854972190 5	.778710528 2	.772914269 1	.772882960 1	.772882959 1
(6), $m = 2 \equiv (7)$	.783878259 7	.772882324 2	.772882959 1	.772882959 1	.772882959 1
(6), $m = 3 \equiv (8)$	.772824071 8	.772882959 1	.772882959 1	.772882959 1	.772882959 1
J.H.He	.783878259 7	.761653645 9	.772888880 8	.772882281 0	.772882959 1
Wei-Wu-Mei	.772824071 8	.772941840 1	.772882959 1	.772882959 1	.772882959 1
(10), $m = 2$	.783878259 7	.761653645 9	.772888880 8	.772882281 0	.772882959 1
(10), $m = 3$	.772824071 8	.772941840 1	.772882959 1	.772882959 1	.772882959 1
(11), $m = 2$	.783878259 7	.761653645 9	.772888880 8	.772882281 0	.772882959 1
(11), $m = 3$	.772824071 8	.772941840 1	.772882959 1	.772882959 1	.772882959 1

As we can see from the previous table, the three methods, the Generalized He-We-Wu-Mei, the improved and the simplified versions, offer the same results, for both $m = 2$ and $m = 3$. This is due to the fact that they started out from the same formula, but by a telescoping reduction of terms, the methods presented by He and Wei-Wu-Mei now have a simple, improved expression, (10). In particular, where $x_0 = x_1$, the recurrence relation is given by (11), (12) which is easier to apply.

However, considering $x_0 = x_1 = 0$, all the methods generate complex values from the first iteration, in the case of $m = 2$. The table below illustrates these inconveniences for the chosen initial data, and also presents the case $m = 5$ for the methods presented in this paper, with an error within 10^{-15} . In this case, the methods offer the exact solution generally after 3 iterations, with the exception of the first one, i.e. the Generalized Newton-Raphson method, that only needs 2 iterations.

Iteration Method	x_2	x_3	x_4
Newton Raphson	1.	.812309030097381	.774276548985500
(6), $m = 2 \equiv (7)$	complex values	complex values	complex values
(6), $m = 3 \equiv (8)$	.767315738099893	.772882959140974	.772882959149210
(6), $m = 5$	.772764910588615	.772882959149210	.772882959149210
J.H.He	complex values	complex values	complex values
Wei-Wu-Mei	.767315738099893	.778393341418516	.772882959270339
(9), $m = 5$	.772764910588615	.773000981896338	.772882959149210
(10), $m = 2$	complex values	complex values	complex values
(10), $m = 3$	.767315738099893	.778393341418516	.772882959270339
(10), $m = 5$	.772764910588615	.773000981896338	.772882959149210
(11), $m = 2$	complex values	complex values	complex values
(11), $m = 2$	.767315738099893	.778393341418516	.772882959270339
(11), $m = 5$	.772764910588615	.773000981896338	.772882959149210

Example

We will now consider the following equation $x - 2 + \ln x = 0$, with the solution $x = 1.5571455989976115131 \dots$. Given the initial values $x_0 = 1$ and $x_1 = 2$, with 5 iterations and an error within 10^{-9} , we obtain

Iteration Method	x_2	x_3	x_4	x_5	x_6
Newton Raphson	1.537901880	1.557098551	1.557145599	1.557145599	1.557145599
(6), $m = 2 \equiv (7)$	1.554445141	1.557145600	1.557145599	1.557145599	1.557145599
(6), $m = 3 \equiv (8)$	1.556698150	1.557145599	1.557145599	1.557145599	1.557145599
J.H.He	1.435675595	1.563093412	1.557012513	1.557145611	1.557145599
Wei-Wu-Mei	1.643227956	1.557324433	1.557146786	1.557145599	1.557145599

(10), m = 2	1.43567559 5	1.56309341 2	1.55701251 3	1.55714561 1	1.55714559 9
(10), m = 3	1.64322795 6	1.55732443 3	1.55714678 6	1.55714559 9	1.55714559 9

The results show anew that the Generalized Newton-Raphson method is the most effective. It offers the numerical solution after only 2 iterations, when $m = 3$, followed by the methods presented by Wei-Wu-Mei and its improved version, which need 4 steps to get to the solution.

We will now add the case $m = 5$ for $x_0 = x_1 = 1.5$. It is shown that all the methods of this order, obtain the solution with an error within 10^{-19} , generally after 3 iterations. Nonetheless, the Generalized Newton-Raphson method proves to be most effective, as for both $m = 5$ and $m = 3$ gets to the solution within the specified precision, after only 2 steps. The results are given below.

Me- tho d \ Itera- tion	x_2	x_3	x_4
N - R	1.556720935135101457 9	1.557145576346602666 7	1.557145598997611291 0
(7)	1.557156517466961487 3	1.557145598997611291 0	1.557145598997611513 1
(8)	1.557145287791697363 4	1.557145598997611513 1	1.557145598997611513 1
(6), m = 5	1.557145598697124322 4	1.557145598997611513 1	1.557145598997611513 1
J.H.He	1.557156517466961487 3	1.557134680558199812 9	1.557145598997611957 2
Wei-Wu-Mei	1.557145287791697363 4	1.557145910203549865 6	1.557145598997611291 0
(9), m = 5	1.557145598697124322 4	1.557145599298098481 7	1.557145598997611513 1
(10), m = 2	1.557156517466961487 3	1.557134680558199812 9	1.557145598997611957 2
(10), m = 3	1.557145287791697363 4	1.557145910203549865 6	1.557145598997611291 0
(10), m = 5	1.557145598697124322 4	1.557145599298098481 7	1.557145598997611513 1
(11), m = 2	1.557156517466961487 3	1.557134680558199812 9	1.557145598997611957 2
(11), m = 3	1.557145287791697363 4	1.557145910203549865 6	1.557145598997611291 0
(11), m = 5	1.557145598697124322 4	1.557145599298098481 7	1.557145598997611513 1

Acceptable results are obtained for each method. Although the fastest methods are obtained in the case $m = 5$, the case $m = 3$ of the Generalized Newton-Raphson method still seems to be the most efficient as its formula is easier to apply. The Wei-Wu-Mei method and the improved one, given in [1], are situated on the second place, in matters of effectiveness. In paper [1], both the improved and the simplified method were obtained by a telescoping reduction of terms using the resolvent equation that generates the Generalized He-Wei-Wu-Mei method. In fact, these three methods use the same equation, as demonstrated by their identical numerical results presented previously for the specified iterative schemes. However, both the improved and the simplified version have a clear advantage regarding the ease of use in the case of elaborate calculus, compared to the Generalized He-Wei-Wu-Mei method.

To conclude, the results presented in the above tables show the importance of considering the polynomial equation (6) obtained in [1], to solve algebraic equations. Besides the fact that it has the simplest form among the proposed resolvent equations, thus requiring fewer calculations, it also provides the fastest convergence to the exact solution. As expected, it also results that the higher the degree of the resolvent equation is, the faster the iteration converges to the exact solution. Particularly, the results show the importance of considering these new methods for solving algebraic equations based on resolvent polynomial equations, instead of the traditional Newton-Raphson method.

4. REFERENCES

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