

ULTRASYMMETRY OF VALUATIONS IN  
TRANSCENDENTAL VALUED FIELDS EXTENSIONSCĂTĂLINA VIȘAN <sup>1\*</sup>**ABSTRACT:**

*Transcendental valued field extensions became an important, but also difficult, area of research lately, due to the complexity of the analysis and characterization of the extensions of their valuations, from  $K$  to  $K(X_1, \dots, X_n)$ , when the transcendental degree is larger than 1. One promising approach introduced the concept of symmetric extensions with respect to the indeterminates  $X_1, \dots, X_n$ , that allowed having a closer look inside the structure of an extension of a valuation from  $K$  to  $K(X_1, \dots, X_n)$ . While these symmetric valuations have been analyzed up to a significant level of detail, a sub-class of them, namely the ultrasymmetric extensions are still an unexplored territory. This paper deals with the investigation of this domain of ultrasymmetric extensions, with the purpose of identifying their form and characteristics.*

**KEYWORDS:** commutative algebra, valued fields, valuations, extensions, symmetry, ultrasymmetry

**Mathematics Subject Classification:** 12F20, 12J10, 13A18

**1. Introduction**

The valuation theory appeared in the last century as a supporting theory for providing a more profound analysis of the  $p$ -adic numbers defined by Kurt Hensel. In a few decades, the theory become very popular among mathematicians, once its valuable applications in number theory, algebraic geometry and functional analysis become apparent. Several papers stand as foundation for this newly developed theory ([1], [2]).

The "general valuation problem", as put by Ostrowski, is the quest of finding a way to construct, given a valued field  $(K, v)$  and a field extension  $L$  of  $K$ , all the possible extensions of the valuation  $v$  on the field  $L$ . While for algebraic extensions the problem has been easily tackled ([3], [4]), for transcendental ones the problem proved to be significantly more difficult. Starting with the simplest case, of extensions from  $K$  to  $K(X)$ , a series of important advances have been made ([5], [6], [7], [8]), culminating with a complete classification of these simplest transcendental extensions in [10], [11], [12] and [13]. Even so, the classification of the general transcendental extensions, from  $K$  to  $K(X_1, \dots, X_n)$  remained a difficult open problem, due to the complicated algebraic geometry involved.

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<sup>1\*</sup> Corresponding author. Assistant Prof., PhD, Romanian-American University, catalina.visan@gmail.com

A special approach of this remaining open problem has been developed in [14] and [15], with the definition of the *symmetric valuations*, that have the advantage of treating the indeterminates  $X_1, \dots, X_n$  consonantly. Being more approachable, these valuations may be used in understanding and solving the general case of extensions from  $K$  to  $K(X_1, \dots, X_n)$ .

This paper intends to continue on this path, by analyzing a sub-category of the symmetric extensions, namely the *ultrasymmetric* ones, sub-category that broadens the concept of symmetry from the values in the value group to the classes of the residue field.

## 2. General notations and definitions

Let  $K$  be a field and  $v$  a valuation on  $K$ . We will use the following notations:

$G_v$  is the value group of  $v$ ;

$O_v$  is the valuation ring of  $v$ ;

$k_v$  is the residue field of  $v$ ;

its  $\rho_v: O_v \rightarrow k_v$  is the residual homeomorphism; for  $x \in O_v$  we denote by  $x^* = \rho_v(x)$  image in  $k_v$ ;

$M_v$  is the maximal ideal of  $v$ .

Two valuations on  $K$ ,  $u$  and  $u'$  are equivalent if there exists an isomorphism of order groups  $j: G_u \rightarrow G_{u'}$  such that  $u' = ju$ ; in this case we write  $u \cong u'$ .

Let  $K'/K$  be an extension of fields. A valuation  $v'$  on  $K'$  is called *an extension* of  $v$  if  $v'(x) = v(x)$  for all  $x \in K$ . If  $v'$  is an extension of  $v$  the residue field  $k_{v'}$  may be identified with a subfield of  $k_v$ , while  $G_{v'}$  may be identified with a subgroup of  $G_v$ .

Let now  $(K, v)$  be a valued field,  $\bar{K}$  an algebraic closure of  $K$  and  $\bar{v}$  an extension of  $v$  to  $\bar{K}$ . Then the residue field of  $\bar{v}$  is an algebraic closure of  $k_v$  and the value group of  $\bar{v}$  is  $\mathbf{Q}G_v$ , the smallest divisible group that still contains  $G_v$ .

We denote by  $K(X)$  the field of rational functions of an indeterminate  $X$  over  $K$  and with  $K[X]$  the ring of polynomials of an indeterminate  $X$  over  $K$ .

An extension,  $u$  of a valuation  $v$  on  $K$  to  $K(X)$  is called *residual-transcendental* (r.t.-extension) if  $k_u / k_v$  is a transcendental extension of fields, *residual-algebraic torsion* (r.a.t.-extension) if  $k_u / k_v$  is algebraic with  $G_u \subseteq \mathbf{Q}G_v$  and *residual-algebraic free* (r.a.f.-extension) if  $k_u / k_v$  is algebraic with  $G_u \not\subseteq \mathbf{Q}G_v$ . The characterization of these extensions may be found in [9].

A *symmetric valuation* (with respect to  $X_1, \dots, X_n$ ) is a valuation  $w$  on  $K(X_1, \dots, X_n)$  such that, given any permutation  $\pi$  of  $\{1, 2, \dots, n\}$  and any  $f \in K(X_1, \dots, X_n)$ , we have

$$w ( f (X_1, X_2, \dots, X_n) ) = w ( f (X_{\pi(1)}, X_{\pi(2)}, \dots, X_{\pi(n)}) ).$$

In this case we denote by  $\pi_f (X_1, X_2, \dots, X_n) = f (X_{\pi(1)}, X_{\pi(2)}, \dots, X_{\pi(n)})$ , the automorphism  $f \rightarrow \pi_f$  of  $K(X_1, \dots, X_n)$  that leaves unchanged the symmetric rational functions of  $K(X_1, \dots, X_n)$ . For a symmetric valuation on  $K(X_1, \dots, X_n)$  we use the following notations for any  $i$ , with  $0 \leq i \leq n$ :

$K_i := K(X_1, \dots, X_i)$ , with the convention  $K_0 = K$ ;

$u_i :=$  the restriction of  $w$  to  $K_i$ , with the conventions  $u_0 = v$ ,  $u_n = w$ ;

$O_i$ ,  $G_i$ , resp.  $k_i :=$  the valuation ring, valuation group, resp. residual field of  $u_i$ ;

An extension  $w$ , of  $v$  from  $K$  to  $K(X_1, \dots, X_n)$ , is called *residual-transcendental simple* (r.t.s.-extension) if there exist  $a \in \overline{K}$  and  $\delta \in \mathbf{Q}G_v$  such that  $w(X_i - X_1) = \delta$ , for all  $i \in \{2, \dots, n\}$  and by denoting:

$g \in K[X]$  the minimal monic polynomial of  $a$ ;

$v'$  an extension of  $v$  la  $K(a)$ ;

$$\gamma := \sum_{\substack{a' \in K \\ g(a')=0}} \inf ( \delta, v'(a' - a) ) ;$$

we get, for any  $F \in K[X_1, \dots, X_n]$  written as:

$$F = \sum_{(i_1, \dots, i_n) \in I} f_{i_1, \dots, i_n} (X_1) \cdot g(X_1)^{i_1} \cdot (X_2 - X_1)^{i_2} \cdot \dots \cdot (X_n - X_1)^{i_n}$$

where  $\deg f_{i_1, \dots, i_n} < \deg g$  and  $I$  is a finite set of  $n$ -uples of indices, the following:

$$w (F) = \inf_{(i_1, \dots, i_n) \in I} \left( v'(f_{i_1, \dots, i_n} (a)) + i_1 \cdot \gamma + (i_2 + \dots + i_n) \cdot \delta \right).$$

The r.t.s.-extensions are a generalization of the Gaussian extensions. A further generalization are the *symmetrically-open* extensions as being those that, when provided any number of new indeterminates,  $X_{n+1}, \dots, X_{n+r}$ , still allow a symmetric extension to  $K(X_1, \dots, X_{n+r})$  with respect to all  $X_1, \dots, X_{n+r}$ .

More details about symmetric valuations may be found in [14] and [15].

We will use the following different measures for a polynomial  $f \in K[X_1, \dots, X_n]$ :

$$\deg f := \max \{ i_1 + \dots + i_n / a \cdot X_1^{i_1} \cdot X_2^{i_2} \cdot \dots \cdot X_n^{i_n} \text{ monomial of } f \};$$

$$\deg_{X_i} f := \text{gradul lui } f \text{ ca polinom \u00een } X_i \text{ peste } K[X_1, X_2, \dots, X_{i-1}, X_{i+1}, \dots, X_n];$$

$\deg^{\wedge} f := \max\{\deg_{X_i} f / i \in \{1, 2, \dots, n\}\}$ , the *covering degree* of  $f$ ;

$\deg_{\vee} f := \min\{\deg_{X_i} f / i \in \{1, 2, \dots, n\}\}$ , the *uncovering degree* of  $f$ ;

$\text{mdeg } f := \sum_{i=1}^n \deg_{X_i} f$ , the *metadegree* of  $f$ ;

being easy to notice that, for any  $i \in \{1, 2, \dots, n\}$ , we have:

$$\deg_{\vee} f \leq \deg_{X_i} f \leq \deg^{\wedge} f \leq \deg f \leq \text{mdeg } f$$

and for  $f, g \in K[X_1, \dots, X_n]$ , all the degrees above may be extended for the rational function  $f/g$  as the maximum between the corresponding degrees of  $f$  and  $g$ :

$$\deg_{\bullet} (f/g) := \max(\deg_{\bullet} f, \deg_{\bullet} g).$$

Denoting by  $e_k^{(n)}$  the symmetrical polynomial of degree  $k$ , we have:

$$\deg_{\vee} e_k^{(n)} = \deg^{\wedge} e_k^{(n)} = 1, \quad \deg e_k^{(n)} = k \quad \text{and} \quad \text{mdeg } e_k^{(n)} = n.$$

## 2. Ultrasymmetric valuations

In paper [15] a sub-class of symmetric valuations was defined and briefly analyzed. This chapter will reiterate the work previously done around ultrasymmetric valuations and add valuable new results in the field.

**Definition 2.1:** A valuation  $w$  on  $K(X_1, \dots, X_n)$ , with  $n \geq 2$ , is called *ultrasymmetric* (with respect to  $X_1, \dots, X_n$ ) if, for any permutation  $\pi \in S_n$  and any  $f \in K(X_1, \dots, X_n)$ , we have:

$$w(f) \geq 0 \Leftrightarrow w(\pi f) \geq 0 \text{ and, when both inequalities are strict, we have } f^* = (\pi f)^* \text{ in } k_w.$$

The first part of the condition above says that all the valuations  $\pi f$  have the same valuation ring, hence they are equivalent. It is easy to verify that they are, in fact, equal ([15; D4.1.1]): suppose this is not true, i.e.  $w$  is ultrasymmetric and, at the same time, there exists  $f \in K(X_1, \dots, X_n)$  such that  $w(f) < w(\pi f)$ . We can assume, without any loss of generality, that  $w(f)$  and  $w(\pi f)$  are minimal with this property among the permutations of  $f$ . Then we would have two cases:

$$(i) \quad w(f) = w(\pi^{-1}f) < w(\pi f), \text{ so } w(f / \pi f) < 0 = w(\pi^{-1}f / f)$$

$$(ii) \quad w(f) < w(\pi f) \leq w(\pi^{-1}f), \text{ so } w(f / \pi f) < 0 < w(\pi f / f) \leq w(\pi^{-1}f / f)$$

but, in both cases, the ultrasymmetry of  $f$  is invalidated, since:

$$w(\pi^{-1}f / f) = w(\pi^{-1}(f / \pi f)).$$

On the other hand, the reciprocal is not true, as shown by the following example. Let  $w$  be the trivial valuation on  $K(X_1, \dots, X_n)$ , with  $n \geq 2$ . It is obviously symmetrical but, taking into consideration that  $k_n$  is isomorphic with  $K_n$ , allowing us to put  $f^* = f$  for any  $f \in K(X_1, \dots, X_n)$ , we get  $X_1^* = X_1 \neq X_2 = X_2^*$  which means that  $w$  cannot be ultrasymmetric. More generally, the following facts were proven.

**Observation 2.2 [15; D4.1.2, D4.1.3]:** A r.t.s.-extension (so, in particular, the Gaussian valuation) with respect to  $X_1, \dots, X_n$ , with  $n \geq 2$ , is not ultrasymmetric.

**Observation 2.3 [15; Corollary 4.5.4]:** A symmetrically-open extension, with respect to  $X_1, \dots, X_n$ , with  $n \geq 3$ , is not ultrasymmetric.

We can now prove the first new result of this paper, a more general aspect regarding the ultrasymmetric valuations, namely the fact that extending a valuation to the field of rational functions in at least four indeterminates, prevents this extension from being ultrasymmetric.

**Proposition 2.4:** Let  $(K, v)$  be a valued field and  $w$  an ultrasymmetric extension of  $v$  to  $K(X_1, \dots, X_n)$ . Then  $n \leq 3$ .

PROOF:

Since  $w$  is ultrasymmetric, it is also symmetric, hence  $w(X_i - X_j) = w(X_{i'} - X_{j'})$  for any indices  $1 \leq i, j, i', j' \leq n$ , with  $i \neq j$  and  $i' \neq j'$ . This means that  $(X_i - X_j) / (X_{i'} - X_{j'})$  is in  $O_w$ . Again, as  $w$  is ultrasymmetric, we have, for any permutation  $\pi \in S_n$ , the following equality in  $k_w$

$$\left( (X_i - X_j) / (X_{i'} - X_{j'}) \right)^* = \left( (X_{\pi(i)} - X_{\pi(j)}) / (X_{\pi(i')} - X_{\pi(j')}) \right)^*$$

where  $1 \leq i, j, i', j' \leq n$ , with  $i \neq j$  and  $i' \neq j'$ .

Let's now put some explicit indices to the indeterminates. Suppose, hypothetically, that we have  $n \geq 4$  and consider  $\pi$  the inversion between 3 and 4. In this case, the equality above may be written:

$$\begin{aligned} 0^* &= \left( (X_1 - X_3) / (X_1 - X_2) \right)^* - \left( (X_1 - X_4) / (X_1 - X_2) \right)^* = \\ &= \left( (X_1 - X_3 - X_1 + X_4) / (X_1 - X_2) \right)^* = \\ &= \left( (X_4 - X_3) / (X_1 - X_2) \right)^*. \end{aligned}$$

Hence, we deduce that  $w(X_4 - X_3) > w(X_1 - X_2)$ , but this contradicts the symmetry of  $w$ .

Q.E.D.

Having this limitation proved, it makes sense to discuss only about  $K(X,Y)$  and  $K(X,Y,Z)$  in the context of ultrasymmetry. First, we will need to establish some exquisite bases for these fields seen as algebraic extensions of the subfields of the corresponding symmetric rational functions. These bases are special in the sense that, for our future convenience, they should have the additional property that all the elements of these bases are units in  $O_w$ . The following proposition proves the existence of such bases and shows how they may be constructed.

**Lemma 2.5:** Let  $(K, v)$  be a valued field and  $w$  a symmetric extension of  $v$  to  $K(X_1, \dots, X_n)$ , with  $2 \leq n \leq 3$ . Let  $K^e(X_1, \dots, X_n)$  be the field of symmetric rational functions of  $K(X_1, \dots, X_n)$ . Then the algebraic extension  $K(X_1, \dots, X_n) / K^e(X_1, \dots, X_n)$  has a basis composed only of units in  $O_w$ . Moreover, when  $w$  is ultrasymmetric, the basis may be chosen such that these units have identical classes in  $k_w$ .

PROOF:

It is a well known fact that the algebraic extension  $K(X_1, \dots, X_n) / K^e(X_1, \dots, X_n)$  has degree  $n!$ . Since  $w$  is symmetric,  $w(X_i) = w(X_j)$  for any indices  $1 \leq i \neq j \leq n$ , which means that all fractions  $X_i / X_j$  are units in  $O_w$ . Obviously, when  $w$  is ultrasymmetric, these fractions also have identical classes in  $k_w$ . The number of fractions  $X_i / X_j$ , with  $1 \leq i \neq j \leq n$ , is  $n(n-1)$ , but we are lucky and  $n(n-1)$  happens to equal  $n!$  when  $2 \leq n \leq 3$ , as required by the hypothesis.

Hence, the only thing that remains to be proved is the fact that  $\{X_i / X_j\}_{1 \leq i \neq j \leq n}$  is a valid basis for  $K(X_1, \dots, X_n) / K^e(X_1, \dots, X_n)$ . In order to prove that, we need five assertions to be verified. For simplicity, we will use the notations  $X := X_1$ ,  $Y := X_2$  and  $Z := X_3$ .

(2.5.1) The element 1 must have a decomposition using  $\{X_i / X_j\}_{1 \leq i \neq j \leq n}$ ; this may be easily checked:

$$\text{for } n = 2, \quad 1 = \frac{X}{Y} \cdot \frac{XY}{X^2 + Y^2} + \frac{Y}{X} \cdot \frac{XY}{X^2 + Y^2};$$

$$\text{for } n = 3, \quad 1 = \frac{X}{Y} \cdot \frac{XYZ}{X^2Z + Y^2X + Z^2Y} + \frac{Y}{Z} \cdot \frac{XYZ}{X^2Z + Y^2X + Z^2Y} + \frac{Z}{X} \cdot \frac{XYZ}{X^2Z + Y^2X + Z^2Y}.$$

(2.5.2) Given the decompositions of  $f, g \in K[X_1, \dots, X_n]$  using  $\{X_i / X_j\}_{1 \leq i \neq j \leq n}$ , there must exist a decomposition of  $f + g$  using the same basis; this assertion is obvious.

(2.5.3) Given the decompositions of  $f, g \in K[X_1, \dots, X_n]$  using  $\{X_i/X_j\}_{1 \leq i \neq j \leq n}$ , there must exist a decomposition of  $fg$  using the same basis; in order to ensure this, we need to prove that any product of two elements of the basis has a decomposition using the same basis; we will check this only for the pairs involving  $X/Y$  and use analogy for the other pairs:

for  $n = 2$ , knowing that 1 has already been proved to have its own valid decomposition, we have:

$$\frac{X}{Y} \cdot \frac{X}{Y} = \frac{X}{Y} \cdot \left( \frac{X}{Y} + \frac{Y}{X} \right) - 1,$$

$$\frac{X}{Y} \cdot \frac{Y}{X} = 1;$$

for  $n = 3$ , by putting  $A := \frac{X}{Y} + \frac{Y}{Z} + \frac{Z}{X}$ ,  $B := \frac{Y}{X} + \frac{Z}{Y} + \frac{X}{Z}$ ,  $C := \frac{XY}{Z^2} + \frac{YZ}{X^2} + \frac{ZX}{Y^2} + 1$ , all being symmetric rational function and knowing that 1 has already been proved to have its own valid decomposition, we have:

$$\frac{X}{Y} \cdot \frac{X}{Y} = \frac{X}{Y} \cdot A - \frac{X}{Z} - \frac{Z}{Y},$$

$$\frac{X}{Y} \cdot \frac{Y}{Z} = \frac{X}{Z},$$

$$\frac{X}{Y} \cdot \frac{Z}{X} = \frac{Z}{Y},$$

$$\frac{X}{Y} \cdot \frac{Y}{X} = 1,$$

$$\frac{X}{Y} \cdot \frac{Z}{Y} = \frac{X}{Y} \cdot \frac{A}{B} + \frac{Y}{X} - \frac{X}{Z} \cdot \frac{2}{B} - \frac{Z}{X} \cdot \frac{A}{B} + \frac{Z}{Y} \cdot \frac{C}{B},$$

$$\frac{X}{Y} \cdot \frac{X}{Z} = \frac{X}{Y} \cdot \left( B - \frac{A}{B} \right) - \frac{Y}{X} + \frac{X}{Z} \cdot \frac{2}{B} + \frac{Z}{X} \cdot \frac{A}{B} - \frac{Z}{Y} \cdot \frac{C}{B} - 1;$$

(2.5.4) Given the decomposition of  $f \in K[X_1, \dots, X_n]$  using  $\{X_i/X_j\}_{1 \leq i \neq j \leq n}$ , there must exist a decomposition of  $1/f$  using the same basis; indeed, we have:

$$\frac{1}{f} = \frac{\prod_{\pi \in S_n, \pi \neq 1_{S_n}} \pi f}{\prod_{\pi \in S_n} \pi f} = \prod_{\pi \in S_n, \pi \neq 1_{S_n}} \left( \pi f \cdot \prod_{\pi \in S_n} \pi f^{-1} \right)$$

which, using (2.5.3) and noticing that  $\prod_{\pi \in S_n} \pi f^{-1}$  is simply a symmetric rational function, leads to a valid decomposition of  $1/f$ ;

(2.5.5) Finally, for any  $i$ , with  $1 \leq i \leq n$ , the element  $X_i$  must have a decomposition using  $\{X_i/X_j\}_{1 \leq i \neq j \leq n}$ ; it is enough to check this for  $X$  and use analogy for the others:

for  $n = 2$ , knowing that  $1$  has already been proved to have its own valid decomposition, we have:

$$X = \frac{X}{Y} \cdot \frac{XY}{2X + 2Y} - \frac{Y}{X} \cdot \frac{XY}{2X + 2Y} + 1 \cdot \frac{X + Y}{2};$$

for  $n = 3$ , knowing that  $\frac{X^2}{YZ}$  has already been proved to have its own valid decomposition, we have:

$$X = \frac{X^2}{YZ} \cdot \frac{XYZ}{X^2 + Y^2 + Z^2} + \frac{Y}{Z} \cdot \frac{XYZ}{X^2 + Y^2 + Z^2} + \frac{Z}{Y} \cdot \frac{XYZ}{X^2 + Y^2 + Z^2}.$$

By proving these five assertions, we proved that  $\{X_i/X_j\}_{1 \leq i \neq j \leq n}$  is a valid basis for the algebraic extension  $K(X_1, \dots, X_n) / K^e(X_1, \dots, X_n)$ .

Q.E.D.

The simplicity of the chosen elements  $b_{i,j}$  for these bases has two major advantages: the fact that  $\text{mdeg } b_{i,j} = 1$  (the minimum possible) and the fact that, as we will see in the next chapter,  $b_{i,j}^* \in \{-1^*, 1^*\}$ .

### 3. Characterization of the ultrasymmetric valuations

We will start with a simple fact about the ultrasymmetric extensions over  $K(X, Y, Z)$ , which represent the most complex type of ultrasymmetric valuations, namely a limitation imposed now on the structure of the residue field.

**Proposition 3.1:** Let  $(K, v)$  be a valued field and  $w$  an ultrasymmetric extension of  $v$  to  $K(X, Y, Z)$ . Then  $\text{char } k_w = 3$ .

PROOF:

Since  $w$  is, implicitly, symmetric, we have  $w(X - Z) = w(Y - Z) = w(X - Y)$ , which means that, by switching  $X$  with  $Y$  and taking ultrasymmetry now into account, we get:

$$\left( \frac{X - Z}{X - Y} \right)^* = \left( \frac{Y - Z}{Y - X} \right)^* \in k_w$$

From this we derive:

$$0 < w\left( \frac{X - Z}{X - Y} - \frac{Y - Z}{Y - X} \right) = w\left( \frac{X + Y - 2Z}{X - Y} \right)$$

so  $w(X + Y - 2Z) > w(X - Y)$ . On other hand, using again the ultrasymmetry of  $w$  and the switching of  $X$  with  $Y$  we get:

$$\left( \frac{X + Z - 2Y}{X + Y - 2Z} \right)^* = \left( \frac{Y + Z - 2X}{X + Y - 2Z} \right)^* \in k_w$$

We may now see that:

$$\begin{aligned} 0 < w\left( \frac{X + Z - 2Y}{X + Y - 2Z} - \frac{Y + Z - 2X}{X + Y - 2Z} \right) &= w\left( 3 \frac{X - Y}{X + Y - 2Z} \right) = \\ &= w(3) + w(X - Y) - w(X + Y - 2Z) < w(3) \end{aligned}$$

which means that  $3^* = 0^*$ , hence  $\text{char } k_w = 3$ .

Q.E.D.

We are ready now to make an important connection between ultrasymmetry and the algebraic degree of the residue field of  $w$  over the residue field of the restriction of  $w$  at the subfield of the symmetric rational functions. What we know already ([3]) is that the extension between these residue fields is also algebraic and its degree is at most  $n!$ . We will discover that ultrasymmetry may impose an even more restrictive limit for this degree.

**Theorem 3.2:** Let  $(K, v)$  be a valued field and  $w$  an extension of  $v$  to  $K(X_1, \dots, X_n)$ . Let  $K^e(X_1, \dots, X_n)$  be the field of symmetric rational functions of  $K(X_1, \dots, X_n)$  and  $k_w^e$  the residual field of the restriction of  $w$  to  $K^e(X_1, \dots, X_n)$ . Then the following statements are true:

(3.2.1) if  $w$  is symmetric and  $\deg(k_w/k_w^e) = 1$  then  $w$  is ultrasymmetric;

(3.2.2) if  $w$  is ultrasymmetric and  $\text{char } K \neq n$  then  $\deg(k_w/k_w^e) = 1$ .

PROOF:

(3.2.1) Suppose  $w$  is symmetric and  $k_w = k_w^e$ . For any  $f \in K(X_1, \dots, X_n)$  with  $f^*$  a class in  $k_w$  we may find  $e \in K^e(X_1, \dots, X_n)$  such that  $f^* = e^*$ . This means that  $w(f - e) > 0$  and, since  $w$  is symmetric, it follows that, given any  $\pi \in S$ , we have

$$w(\pi f - e) = w(f - e) > 0$$

which means that  $(\pi f)^* = e^*$ , hence  $(\pi f)^* = f^*$ . We proved, thus, that  $w$  is ultrasymmetric.

(3.2.2) Now suppose  $w$  is ultrasymmetric and  $\text{char } K \neq n$ . Then, according to Proposition 2.4,  $n \leq 3$  so we have to analyze two different cases. For simplicity, we will use the notations  $X := X_1$ ,  $Y := X_2$  and  $Z := X_3$ .

First, let  $n = 2$  and  $F \in K(X, Y)$  with  $F^*$  a class in  $k_w$ . Using Lemma 2.5, we write  $F$  as:

$$F = \frac{X}{Y} \cdot f + \frac{Y}{X} \cdot g, \text{ with } f, g \in K^e(X, Y)$$

Let  $e = (f + g) / 2$ , which exists since  $\text{char } K \neq 2$ . We have, using the ultrasymmetry of  $w$ :

$$\begin{aligned} 0 < w\left(\left(\frac{X}{Y} \cdot \frac{f}{2} + \frac{Y}{X} \cdot \frac{g}{2}\right) - \left(\frac{X}{Y} \cdot \frac{g}{2} + \frac{Y}{X} \cdot \frac{f}{2}\right)\right) &= \\ &= w\left(\frac{X}{Y} \cdot (f - e) + \frac{Y}{X} (g - e)\right) = \\ &= w\left(F - \frac{X^2 + Y^2}{XY} \cdot e\right) \end{aligned}$$

which means that  $F^* = \left(\frac{X^2 + Y^2}{XY} \cdot e\right)^* \in k_w^e$ .

Second, let  $n = 3$  and  $F \in K(X, Y, Z)$  with  $F^*$  a class in  $k_w$ . Using again Lemma 2.5, we write  $F$  as:

$$F = \frac{X}{Y} \cdot f + \frac{Y}{Z} \cdot g + \frac{Z}{X} \cdot h + \frac{Y}{X} \cdot r + \frac{Z}{Y} \cdot s + \frac{X}{Z} \cdot t, \text{ with } f, g, h, r, s, t \in K^e(X, Y, Z)$$

Let  $d = (f + g + h) / 3$  and  $e = (r + s + t) / 3$ , which exist since  $\text{char } K \neq 3$ . We have, using again the ultrasymmetry of  $w$ :

$$\begin{aligned}
 0 &< w\left(\left(\frac{X}{Y} \cdot (d-g) + \frac{Y}{Z} \cdot 0 + \frac{Z}{X} \cdot (h-d) + \frac{Y}{X} \cdot (e-s) + \frac{Z}{Y} \cdot 0 + \frac{X}{Z} \cdot (t-e)\right) - \right. \\
 &\quad \left. - \left(\frac{X}{Y} \cdot (h-d) + \frac{Y}{Z} \cdot (d-g) + \frac{Z}{X} \cdot 0 + \frac{Y}{X} \cdot (t-e) + \frac{Z}{Y} \cdot (e-s) + \frac{X}{Z} \cdot 0\right)\right) = \\
 &= w\left(\frac{X}{Y} \cdot (f-d) + \frac{Y}{Z} \cdot (g-d) + \frac{Z}{X} \cdot (h-d) + \frac{Y}{X} \cdot (r-e) + \frac{Z}{Y} \cdot (s-e) + \frac{X}{Z} \cdot (t-e)\right) = \\
 &= w\left(F - \frac{X^2Z + Y^2X + Z^2Y}{XYZ} \cdot d - \frac{Y^2Z + Z^2X + X^2Y}{XYZ} \cdot e\right)
 \end{aligned}$$

which means that  $F^* = \left(\frac{X^2Z + Y^2X + Z^2Y}{XYZ} \cdot d + \frac{Y^2Z + Z^2X + X^2Y}{XYZ} \cdot e\right)^* \in k_w^e$ .

Since, in both cases,  $F$  was arbitrarily chosen, it follows that  $k_w = k_w^e$ . **Lemma 2.5**

Q.E.D.

To be noted that the two statements above are almost reciprocal; in fact, ultrasymmetry and symmetry plus identical residue fields are equivalent except the peculiar case when  $K$  has a characteristic equal to the number of indeterminates. One other remark is that, in the case  $n = 3$ , the residue field  $k_w$  obeys, qualitatively, the exact opposite condition that was imposed on  $K$  in order to have  $k_w = k_w^e$ .

#### 4. Conclusion

We studied the ultrasymmetric extensions of a valuation  $v$  on  $K$  to  $K(X_1, \dots, X_n)$ , from the perspective of the characteristics of the fields  $K$  and  $k_w$  and the degree of the algebraic extension  $k_w / k_w^e$ . We defined a convenient basis of the field extension  $K(X_1, \dots, X_n) / K^e(X_1, \dots, X_n)$  whose elements have minimal metadegree and their classes are  $\pm 1^*$ .

As a consequence of the results presented in this paper, the following table shows the complete classification of the ultrasymmetric valuations:

| # | $n$ | characteristics                                   | relations with $w(X)$            | class $(X/Y)^*$ | comments      |
|---|-----|---------------------------------------------------|----------------------------------|-----------------|---------------|
| 1 | 2   | $\text{char } k_w \neq n$                         | $w(X-Y) > w(X+Y) = w(X)$         | $1^*$           | $k_w = k_w^e$ |
| 2 | 2   | $\text{char } k_w \neq n$                         | $w(X+Y) > w(X-Y) = w(X)$         | $-1^*$          | $k_w = k_w^e$ |
| 3 | 2   | $\text{char } k_w = n$<br>$\text{char } K \neq n$ | $w(X-Y) > w(X+Y) = w(2X) > w(X)$ | $-1^* = 1^*$    | $k_w = k_w^e$ |

|   |   |                                   |                                                      |              |                                                             |
|---|---|-----------------------------------|------------------------------------------------------|--------------|-------------------------------------------------------------|
| 4 | 2 | char $k_w = n$<br>char $K \neq n$ | $w(X + Y) > w(X - Y) =$<br>$= w(2X) > w(X)$          | $-1^* = 1^*$ | $k_w = k_w^e$                                               |
| 5 | 2 | char $k_w = n$<br>char $K \neq n$ | $w(2X) \geq w(X - Y) =$<br>$= w(X + Y) > w(X)$       | $-1^* = 1^*$ | $k_w = k_w^e$                                               |
| 6 | 2 | char $K = n$                      | $\infty = w(2X) > w(X - Y) =$<br>$= w(X + Y) > w(X)$ | $-1^* = 1^*$ | -                                                           |
| 7 | 3 | char $k_w = n$<br>char $K \neq n$ | $w(X - Y) > w(X + Y) = w(X)$                         | $1^*$        | $k_w = k_w^e$ ,<br>restriction to $K(X, Y)$<br>is of type 1 |
| 8 | 3 | char $K = n$                      | $w(X - Y) > w(X + Y) = w(X)$                         | $1^*$        | restriction to $K(X, Y)$<br>is of type 1                    |

## 5. References

- [1] O. F. G. Schilling, **The theory of valuations**, A. M. S. Math. Surveys, Nr. 4. Providence, Rhode Island, 1950
- [2] P. Samuel, O. Zariski, **Commutative Algebra**, Vol. II, D. Van Nostrand, Princeton, 1960
- [3] N. Bourbaki, **Algebre Commutative**, Herman. Paris, (1964)
- [4] O. Endler, **Valuation Theory**, Springer, Berlin-Heidelberg-New York, 1972
- [5] J. Ohm, **Simple transcendental extensions of valued fields**, J. Math. Kyoto Univ. 22 (1982), 201-221
- [6] V. Alexandru, N. Popescu, **Sur une classe de prolongements à  $K(X)$  d'une valuation sur une corp  $K$** , Rev. Roumaine Math. Pures Appl., 5 (1988), 393-400
- [7] V. Alexandru, N. Popescu, A. Zaharescu, **A theorem of characterization of residual transcendental extensions of a valuation**, J. Math. Kyoto Univ. 24-8 (1988) 579-592
- [8] V. Alexandru, N. Popescu, A. Zaharescu, **Minimal pairs of definition of a residual transcendental extension of a valuation**, J. Math. Kyoto Univ. 30-2 (1990) 207-225
- [9] V. Alexandru, N. Popescu, A. Zaharescu, **All valuations on  $K(x)$** , J. Math. Kyoto Univ. 30-2 (1990) 281-296
- [10] S. K. Khanduja, **On valuations of  $K(x)$** , Proceedings of the Edinburgh Mathematical Society. 35 (1992) 419-426

- [11] N. Popescu, A. Zaharescu, **On a class of valuations on  $K(\mathbf{x})$** , *11th National Conference of Algebra, Constanța, 1994, An. St. Univ. Ovidius Constanța, Seria: Mat. Vol. II (1994)*, 120-136. [53]
- [12] N. Popescu, A. Zaharescu, **On the structure of the irreducible polynomials over local fields**, *J. Number Theory*, 52, No. 1, 1995, pp. 98-118
- [13] N. Popescu, C. Vraciu, **On the extension of a valuation on a field  $K$  to  $K(X)$** , *Rendiconti del Seminario Matematico della Università di Padova*, 96 (1996), p. 1-14
- [14] C. Vișan, **Symmetric Extensions of a Valuation on a Field  $K$  to  $K(X_1, \dots, X_n)$** , *International Journal of Algebra*, Vol. 6, 2012, no. 26, 1273 - 1288
- [15] C. Vișan, **Characterization of Symmetric Extensions of a Valuation on a Field  $K$  to  $K(X_1, \dots, X_n)$** , *Annals of the University of Bucharest*, Vol. 6 (LXIV), no. 1, 2015, 119-146