NUMERICAL METHODS FOR SOLVING SDES CASE STUDY

Ovidiu Solomon[1]

ABSTRACT

The Euler - Maruyama and Milstein methods are applied to approximate the solution of linearly Langevin equation with multiplicative noise. The exact solution is obtained by applying the Ito's lemma. It is worth mentioning that not always the discretization used to find the solutions of SDEs (Stochastic Differential Equations) leads to a convenient convergence when the discretization step tends to zero.

Keywords: stochastic differential equation, numerical methods, Langevin equation, multiplicative noise

1. INTRODUCTION

The concept of stochastic differential equations was introduced for the first time in 1902 by Gibbs when he studied the integral of Hamilton-Jacobi differential equations to conserve systems in statistical mechanics with random initial states [1]. A rigorously description in terms of mathematical language was made in 1951 by Ito [2]. The descriptions of solutions for stochastic differential equations were attempted by the integration of Fokker Planck equations. Based on the Euler classical method used in case of ordinary differential equations, the Euler- Maruyama method finds a solution by determining a realization of the stochastic process [3]. The characterization of methods for numerical integration was made by Clark and Cameron [4]. Some theoretical and practical investigations in this field were made in [14]-[30].

It is considered the Ito stochastic equation [5]:

$$\mathrm{d}x(t) = f(x(t),t)\mathrm{d}t + g(x(t),t)\mathrm{d}w(t),\ \ x(0) = x_0,\ \ 0 \le t \le \mathrm{T}. \tag{1}$$

where f and g are real functions, $x(0)$ is a random variable and $w(t)$ is a Wiener process and $d\mathrm{w}(t) = w(t+dt) - w(t)$ is the random process of infinitesimal increases of $w(t)$ with proprieties

$$E[d\mathrm{w}(t)] = 0; E[\left(d\mathrm{w}(t)\right)^2] = dt; E[d\mathrm{w}(t)d\mathrm{w}(t')] = 0,\ t \ne t'.$$

[1] Assistant Teacher, Ph.D., Romanian-American University, Bucharest, Solomon.ovidiu@profesor.rau.ro

The following theorem presents the conditions under Itô stochastic equation (1) admits the solution and it is singular and they all these aspects are presented in the following theorem [5]:

Theorem 1

Suppose the functions f şi g and the initial condition x_0 satisfy the hypotheses [2]: H_1: There is a K such that:

$$|f(x,t)| \leq K(1+|x|^2)^{1/2}, |g(x,t)| \leq K(1+|x|^2)^{1/2}, \quad x \in \square \; ; \tag{2}$$

H_2: f and g satisfy the uniform Lipschitz conditions in x:

$$|f(x_2,t) - f(x_1,t)| \leq K|x_2 - x_1|, |g(x_2,t) - g(x_1,t)| \leq K|x_2 - x_1|; \tag{3}$$

H_3: f and g satisfy the uniform Lipschitz conditions in t:

$$|f(x_2,t) - f(x_1,t)| \leq K|x_2 - x_1|, |g(x_2,t) - g(x_1,t)| \leq K|x_2 - x_1|; \tag{4}$$

H_4: x_0 is a random variable with $E[|x_0|^2] < \infty$ independent of $dw(t),\ t \in [0,T]$.

Then, the integral equation

$$x(t) - x_0 = \int_0^t f(x(\tau))d\tau + \int_0^t g(x(\tau))dw(\tau), \quad t \in [0,T] \tag{5}$$

has a solution on $[0,T]$ in the mean square sense with the properties:

P_1: $x(t)$ is mean square continuous on $[0,T]$; (6)

P_2: $E[|x(t)|^2] < \infty$ on $[t_0,T]$ (7)

P_3: $\int_{t_0}^{T} E[|x(t)|^2]dt < \infty$; (8)

P_4: $x(t) - x_0$ is independent of $\{dw(\tau),\ \tau \geq t\},\ \forall t \in [0,T]$. (9)

P_5: $x(t)$ is a Markov process, and in the mean square sense it is uniquely determined by the initial condition x_0 (10)

The existence proof is made by successive approximations in [5].

2. THE LANGEVIN EQUATION

The bombardment intensity of pollen particles suspended in water with water molecules does not depend on the state variables, i.e. the particle position and velocity. Using the notation $x(t)$ for the velocity of such particles, Langevin wrote the equation [6], [7]:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t) + bz(t) \tag{11}$$

for the velocity of the particle, where molecular forces are given by white noise $z(t)$, and a, b are constants. One can interpret the Langevin equation (11) symbolically like a stochastic differential equation

$$\mathrm{d}x(t) = ax(t)\mathrm{d}t + b\mathrm{d}w(t) \tag{12}$$

leading to a stochastic integral equation

$$x(t) = x(0) + \int_0^t ax(s)\mathrm{d}s + \int_0^t b\mathrm{d}w(s) \tag{13}$$

where the second is an Ito integral. With external fluctuations, the intensity of the noise depends on the state of the system. For example [7], the growth coefficient from exponentially growth equation $\dot{x} = \alpha x$ may vary due to environmental effects $\alpha = a + bz(t)$, where a, b are positive constants, and $z(t)$ is the white noise. Thus, it can be written as [7]

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t) + bx(t)z(t) \tag{14}$$

that turns into stochastically differential equation

$$\mathrm{d}x(t) = ax(t)\mathrm{d}t + bx(t)\mathrm{d}w(t) \tag{15}$$

or equivalent in stochastically integral equation

$$x(t) = x(0) + \int_0^t ax(s)\mathrm{d}s + \int_0^t bx(s)\mathrm{d}w(s) \tag{16}$$

Technically, random forcing from (11-13) is called additive noise, and in (14-16) is called multiplicative noise. Generally, both cases are included in the following stochastically differential form:

$$\mathrm{d}x(t) = a\big(x(t)\big)\mathrm{d}t + b\big(x(t)\big)\mathrm{d}w(t) \tag{17}$$

or equivalent integral form

$$x(t) = x(0) + \int_0^t a\big(x(s)\big)\mathrm{d}s + \int_0^t b\big(x(s)\big)\mathrm{d}w(s) \tag{18}$$

Using the Ito lemma [8], the solution of equations (14) – (16) is

$$x(t) = x(0)\exp\left[\left(a - \frac{1}{2}b^2\right)t + bw(t)\right] \tag{19}$$

3. THE EULER-MARUYAMA AND MILSTEIN METHODS

To apply the numerical method for solving the equation (1) on [0, T] we discretize this interval taking $\delta t = \mathrm{T}/\mathrm{L}$ for positive and integer L and $\tau_j = j\delta t$. Numerical approximation $y(\tau_j)$ will be noted by y_j. Equation (17) is transformed through the Euler – Maruyama method in

$$\begin{aligned} &y_j = y_{j-1} + f(y_{j-1})\Delta t + g(y_{j-1})(w(\tau_j) - w(\tau_{j-1})), \;\; j = 1,2,\ldots,\mathrm{L} \\ &y_0 = x_0 \end{aligned} \tag{20}$$

and through the Milstein method [10] in

$$\begin{aligned} y_j = y_{j-1} &+ f(y_{j-1})\Delta t + g(y_{j-1})(w(\tau_j) - w(\tau_{j-1})) + \\ &+ \frac{1}{2}g(y_{j-1})g'(y_{j-1})\Big((w(\tau_j) - w(\tau_{j-1}))^2 - \Delta t\Big), \;\; j = 1,2,\ldots,\mathrm{L}. \\ y_0 = x_0 & \end{aligned} \tag{21}$$

The recursive formulas (20) and (21) provide approximate values in discrete time values τ_j and for intermediate values it can be made an interpolation of them.

It can be seen that the equation (1) can be written

$$x(t_j) = x(t_{j-1}) + \int_{\tau_{j-1}}^{\tau_j} f(x(s))\mathrm{d}s + \int_{\tau_{j-1}}^{\tau_j} g(x(s))\mathrm{d}w(s). \tag{22}$$

Every right term of the equations (20) and (21) approximate the corresponding term of the equation (22). It is noted that if g = 0 and x_0 is a constant then (20) and (21) are reduced to the well-known Euler method.

The convergence criteria are imposing when it's tried to analyze how the exact solution $x = \{x(t),\ t \in [0,T]\}$ is approximated by the numerical solution $y = \{y_j,\ j = 0,1,2,...,N\}$. Since the numerical solution starts from the same initial value, it can be considered that the approximation errors are accumulated in time, so we analyze the approximation accuracy when comparing the difference in the final moment

$$\varepsilon(\delta) = E\left[|x_T - y_N|\right] \le \left\{E\left[|x_T - y_N|^2\right]\right\}^{1/2}. \tag{23}$$

The process $y = \{y_j,\ j = 0,1,2,...,N\}$ strong converge with order $\gamma \in (0,\infty)$ through exact solution $x = \{x(t),\ t \in [0,T]\}$ if there is a positive and bounded constant K and a positive constant δ_0 so that

$$\varepsilon(\delta) \le K\delta^{\gamma} \tag{24}$$

for every time discretization with maximum step $\delta \in (0,\delta_0)$.

The convergence strong order of approximate process in case of Euler – Maruyama method is $\gamma = 0,5$ (due to increments Δw_j which have the order of convergence $\delta^{1/2}$ and not δ), that is lower than the deterministic case when it has value 1 and $\gamma = 1$ in case of the Milstein method. In cases where we are interested just for the mean square or mean square of the final moments, we use the weak convergence criterion, comparing the various orders of the final moments between the exact solution and the final solution. If the differential equation (1) admits the solution and it is singular for the given initial conditions, then we can determine the nature of convergence depending on the discretization step chosen based on several numerical approximations calculated for various steps δ [7].

Regarding the numerical part, it is chosen the step δt multiples of $R \geq 1$ for the increment Δt from Brownian motion. This guarantees that the set of points $\{t_j\}$ underlying the discrete Brownian trajectory contains the points $\{\tau_j\}$ of the Euler–Maruyama and the Milstein methods. If there is an analytical way of solving stochastic differential equation then can use an arbitrary step δt. The following theorem shows that the Euler - Maruyama method strong converges to the exact solution with order $\gamma = 0,5$ and the Milstein approximation strong converges to the exact solution with order $\gamma = 1$ [7], [9].

Theorem 2

The roughly numerical solution of the Ito equation depending on the discrete step δ is denoted by $\{y_{\tau_j}^{\delta}\} = \{y_j^{\delta},\ j = 0,1,2,...,N\}$. We suppose that [7]:

$$
\begin{aligned}
&\mathrm{E}\left[|x_0|^2\right] < \infty \\
&\left\{\mathrm{E}\left[\left|x_0 - y_0^{\delta}\right|^2\right]\right\}^{1/2} \leq K_1\sqrt{\delta} \\
&\left|f(t,x) - f(t,y)\right| + \left|g(t,x) - g(t,y)\right| \leq K_2\left|x - y\right| \\
&\left|f(t,x)\right| + \left|g(t,x)\right| \leq K_3\left(1 + |x|\right) \\
&\left|f(s,x) - f(t,y)\right| + \left|g(s,x) - g(t,y)\right| \leq K_4\left(1 + |x|\right)\sqrt{|s-t|}
\end{aligned}
\tag{25}
$$

for every $s,t \in [t_0,t]$ and $x, y \in \mathbb{R}$, K_1, K_2, K_3, K_4 does not depend on δ.

Thus, we have that

$$\mathrm{E}\left[\left|x_T - y_N^{\delta}\right|^2\right] < K_5\sqrt{\delta}\ . \tag{26}$$

for the Euler – Maruyama approximation and

$$\mathrm{E}\left[\left|x_T - y_N^{\delta}\right|^2\right] < K_5\delta\ . \tag{27}$$

for the Milstein approximation, where the constant K_5 does not depend on δ [7].

4. NUMERICAL APPLICATION OF METHODS

We apply the Euler - Maruyama and the Milstein methods for linear stochastic differential equation (15) with exact solution (19) where $a = 2,\ b = 1,5$. The exact solution is graphically illustrated through a continuous black line (-). We considered $x_0 = 1$, the interval [0, 1] and $\Delta t = 2^{-8}$. For numerical simulation through the Euler – Maruyama It is considered $\delta t = R\Delta t$ and the first three natural values of *R*: 1, 2 and 3. The methods requires the necessity of using the increment $w(\tau_j) - w(\tau_{j-1})$ given by the relation [8], [11]:

$$w(\tau_j) - w(\tau_{j-1}) = w(jR\delta t) - w((j-1)R\delta t) = \sum_{k=jR-R+1}^{jR} dw_k\ . \qquad (28)$$

The following figures shows the exact solution and the numerical solution obtained through the methods the Euler - Maruyama (-) and the Milstein (*) for different values of R (multiple increment of Δt for the Brownian motion) [11]-[13].

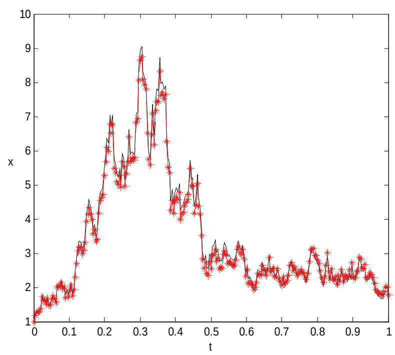

Fig.1. Exact solution (-) and the numerical solution (*) obtained through the Euler – Maruyama method (R = 1)

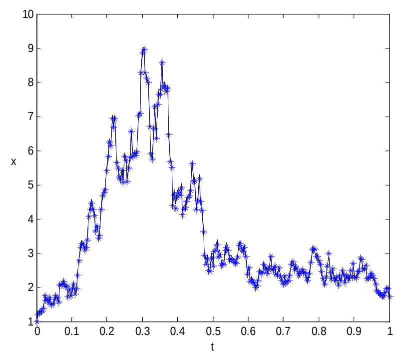

Fig. 2. Exact solution (-) and the numerical solution (*) obtained through the Milstein method (R = 1)

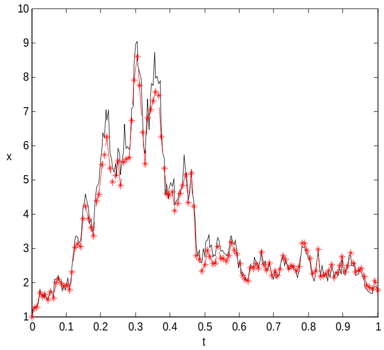

Fig. 3. Exact solution (-) and the numerical solution (*) obtained through the Euler – Maruyama method (R = 2)

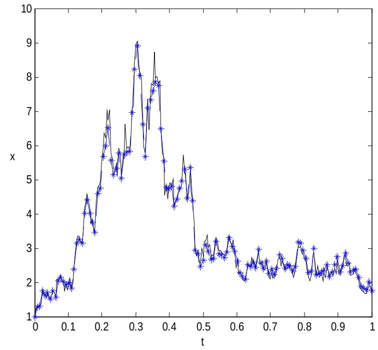

Fig. 4. Exact solution (-) and the numerical solution (*) obtained through the Milstein method (R = 2)

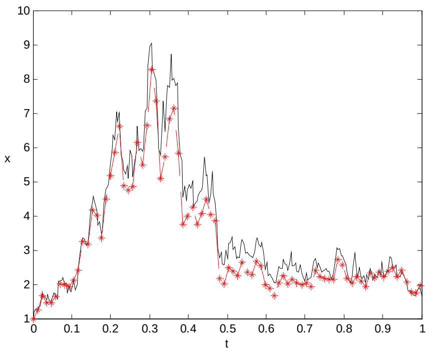

Fig. 5 Exact solution (-) and the numerical solution (*) obtained through the Euler – Maruyama method (R = 3)

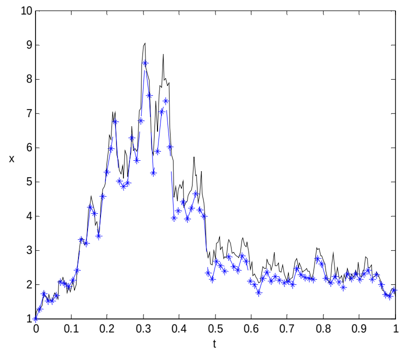

Fig. 6. Exact solution (-) and the numerical solution (*) obtained through the Milstein method (R = 3)

The solution given by the Euler – Maruyama method is represented by a dotted red line (-), and that provided by the Milstein method by a dotted blue line (*). It is denoted with *emerr* the difference between the exact solution and the numerical solution obtained by Euler – Maruyama method at time t = T and with *merr* the difference between the exact solution and the numerical solution obtained by Milstein method at time t = T. Taking $\delta t = R\Delta t$ with values of R 1, 2 and then 3, we have the following table:

Table 1

R	emerr	merr
1	0,126	0,077
2	0,141	0,108
3	0,306	0,195

5. CONCLUSIONS

We applied the Euler-Maruyama and Milstein methods for numerical solving of a particular SDE (linearly Langevin equation with multiplicative noise), first applying Ito lemma.

It can be seen that not always the discretization used to approximate the solutions of stochastic differential equations leads to a convenient convergence to a solution when the maximum discretization step tends to zero [7], [9].

In our case, the Milstein method offers a good approximation than Euler method referring to difference between exact solution and numerical solution obtained by these methods.

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