

USE OF SPECTRAL THEORY OF MATRICES TO STUDY SEISMIC MOVEMENTS

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Abstract

To the worldwide efforts made for understand the dynamics of seismic movements, a significant contribution was made by the research group of Massachusetts Institute of Technology. Such a contribution was given in 1983 by L. R. Lines and S. Treitel, [4]. Starting from ideas contained in this paper [4], we present here, in all mathematical details, how the stages of development of earthquake can be characterized. The impulse response of the main filter that characterizes the seismic movement is obtained by minimizing a second moment norm. This is made by Lagrange multipliers method. The obtained impulse responses are found to be eigenvectors of some matrix, named moment of inertia matrix. The properties of this matrix are specified. A simple example to emphasize the theory, including the deduction of relations between the main parameters of the earthquake is given, using discrete convolution and deconvolution. Several conclusions are finally presented.

Keywords: earthquake, second moment norm minimization, Lagrange multipliers method, eigenvalues and eigenvectors of a matrix.

1. Theoretical foundations

The discrete seismic trace measured on the surface of the Earth will be referred as x_i , where $i = 0, 1, \dots, t$ represents the discrete time. This movement is the output of the time-invariant physical system, that is determined by the input gives by the seismic trace w_i , produced in depth, the system having the impulse response f_i given by the layered medium that is passed by the wavelet. We make the convention that $w_i = 0$, if $t < 0$, or $t > T$.

Therefore, the relationship between these two measures is given by the discrete convolution product.

$$x_i = \sum_{j=0}^t f_j w_{i-j} \cdot (1)$$

This will be noted as:

$$(x_i) = (f_i) * (w_i), \quad (1)$$

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The estimate \hat{f}_i of f_i is measured at the output of a *noise filter* when the input is x_i . The filter's impulse response is denoted with $a = (a_i : i = 0, 1, \dots, t)$

Therefore the desired estimate is given by:

$$\hat{f}_i = \sum_{n=0}^t a_n x_{i-n}. \quad (2)$$

Written also as:

$$(\hat{f}_i) = (a_i) * (x_i), \quad (2')$$

By substituting (1) in (2) we obtain the estimate \hat{f}_i of f_i to be:

$$\hat{f}_i = \sum_{n=0}^t a_n \sum_{j=0}^t f_j w_{i-n-j} = \sum_{j=0}^t f_j \sum_{n=0}^t a_n w_{i-j-n} = \sum_{j=0}^t f_j S_{i-j} \quad (3)$$

That is the output at the input f_i through the physical system:

$$(\hat{f}_i) = (S_i) * (f_i), \quad (3')$$

with the impulse response given by:

$$S_i = \sum_{n=0}^t a_n w_{i-n}, \quad (4)$$

That can also be written as:

$$(S_i) = (a_i) * (w_i) \quad (4')$$

The function (S_i) is named *the resolving kernel* or *scanning function*.

The noise filter's impulse response a is supposed to have a constant power $\|a\|$, hence

$$\|a\|^2 = \sum_{n=0}^t a_n^2 = C,$$

Therefore, the following condition must be satisfied:

$$q = C - \sum_{n=0}^t a_n^2 = 0, \quad (5)$$

where C is a constant.

The components a_0, a_1, \dots, a_t of the noise filter's impulse response a will be determined so that *the second moment norms*:

$$q_i = \sum_{j=0}^t (i-j)^2 S_j^2 = \sum_{j=0}^t (i-j)^2 \left(\sum_{n=0}^t a_n w_{j-n} \right)^2, \quad (6)$$

of the resolving kernel (S_i) to be minimal. The minimum value of the functions q_i given by the formula (6), having the variables a_i , is a conditioned minimum with restriction (5). It will be solved using the Lagrange multipliers method.

For this we form the function:

$$Q_i = q_i + \lambda q = \sum_{j=0}^t (i-j)^2 \left(\sum_{n=0}^t a_n w_{j-n} \right)^2 + \lambda \left(C - \sum_{n=0}^t a_n^2 \right), \quad i = 0, 1, \dots, t.$$

(7)

The variables a_0, a_1, \dots, a_t and λ will be determined from the conditions:

$$\begin{aligned} \frac{\partial Q_i}{\partial a_m} &= \frac{\partial q_i}{\partial a_m} + \lambda \frac{\partial q}{\partial a_m} = 2 \sum_{j=0}^t (i-j)^2 w_{j-m} \sum_{n=0}^t a_n w_{j-n} - 2\lambda a_m = \\ &= 2 \sum_{n=0}^t a_n \sum_{j=0}^t (i-j)^2 w_{j-m} w_{j-n} - 2\lambda a_m = 0, \quad m = 0, 1, \dots, t. \end{aligned}$$

(8) and:

$$\frac{\partial Q_i}{\partial \lambda} = C - \sum_{n=0}^t a_n^2 = 0.$$

The relation (8) has the form:

$$\sum_{n=0}^t a_n F_{n,m}(i) = \lambda a_m, \quad i, m = 0, 1, \dots, t,$$

(9)

if we denote

$$F_{n,m}(i) = \sum_{j=0}^t (i-j)^2 w_{j-m} w_{j-n}, \quad i, n, m = 0, 1, \dots, t.$$

(10)

Let's observe that in the sum from (10) we will obtain zero if $j = i$, or $j < n$ or $j < m$, so we will obtain zero when $j < \max(m, n)$ and when $j > n + T$ and $j > m + T$. Therefore, we need to choose $t = \min(m, n) + T$, because for every index $j > \min(m, n) + T$ the corresponding terms are null.

The relation (9) can be put in the matrix form:

$$F(i)A(i) = \lambda(i)A(i), \quad i = 0, 1, \dots, t$$

(11)

where

$$F(i) = [F_{n,m}(i)] = \begin{bmatrix} F_{0,0}(i) & F_{0,1}(i) & \cdots & F_{0,t}(i) \\ F_{1,0}(i) & F_{1,1}(i) & \cdots & F_{1,t}(i) \\ \cdots & \cdots & \cdots & \cdots \\ F_{t,0}(i) & F_{t,1}(i) & \cdots & F_{t,t}(i) \end{bmatrix}, \quad i = 0,1,\dots,t,$$

(12)

is a square matrix of order $t+1$, and:

$$A(i) = [a_0(i) \ a_1(i) \ \cdots \ a_t(i)]',$$

(13)

the accent representing the transpose of the row matrix, the column matrix being the impulse response of the noise filter. The formula (11) shows that at every time moment i the variable $\lambda(i)$ is an eigenvalue of the matrix $F(i)$, which is named *the moment of inertia matrix*. The column matrices $A(i) = [a_0(i) \ a_1(i) \ \cdots \ a_t(i)]'$ are the eigenvectors of the moment of inertia matrix corresponding to those eigenvalues.

2. Properties of the moment of inertia matrix

The associated matrix $F(i)$ is symmetrical, because $F_{n,m}(i) = F_{m,n}(i)$, $i, n, m = 0,1,\dots,t$, and also positive defined, therefore its eigenvalues λ are positive or null real numbers.

Indeed, we consider

$$\begin{aligned} J(i) &= A(i)' F(i) A(i) = [a_0(i) \ a_1(i) \ \cdots \ a_t(i)] \begin{bmatrix} F_{0,0}(i) & F_{0,1}(i) & \cdots & F_{0,t}(i) \\ F_{1,0}(i) & F_{1,1}(i) & \cdots & F_{1,t}(i) \\ \cdots & \cdots & \cdots & \cdots \\ F_{t,0}(i) & F_{t,1}(i) & \cdots & F_{t,t}(i) \end{bmatrix} \begin{bmatrix} a_0(i) \\ a_1(i) \\ \vdots \\ a_t(i) \end{bmatrix} = \\ &= [a_0(i) \ a_1(i) \ \cdots \ a_t(i)] \begin{bmatrix} \sum_{m=0}^t a_m(i) F_{0,m}(i) \\ \sum_{m=0}^t a_m(i) F_{1,m}(i) \\ \vdots \\ \sum_{m=0}^t a_m(i) F_{t,m}(i) \end{bmatrix} = \sum_{n=0}^t \sum_{m=0}^t a_n(i) a_m(i) F_{n,m}(i) = \\ &= \sum_{n=0}^t \sum_{m=0}^t a_n(i) a_m(i) \sum_{j=0}^t (i-j)^2 w_{j-n} w_{j-m} = \\ &= \sum_{j=0}^t \sum_{n=0}^t (i-j) a_n(i) w_{j-n} \sum_{m=0}^t (i-j) a_m(i) w_{j-m} = \sum_{j=0}^t G_j^2(i) \geq 0, \quad i = 0,1,\dots,t, \end{aligned}$$

where:

$$G_j(i) = \sum_{n=0}^t (i-j)a_n(i)w_{j-n}, \quad i, j = 0, 1, \dots, t.$$

By multiplying relation (11) with the matrix $A(i)'$ we obtain:

$$J(i) = A(i)' F(i) A(i) = \lambda(i) A(i)' A(i) = \lambda(i) \sum_{n=0}^t a_n(i)^2 = \lambda(i) \|a(i)\|^2,$$

hence it results:

$$\lambda(i) = \frac{J(i)}{\|a(i)\|^2} \geq 0, \quad i = 0, 1, \dots, t.$$

3. Application

We consider the wavelet $(w_0, w_1) = (2, 1)$ with $T = 1$. At the moment $i = 0$, according to (10) we have:

$$F_{n,m}(0) = \sum_{j=0}^t j^2 w_{j-n} w_{j-m} = \sum_{j=1}^t j^2 w_{j-n} w_{j-m}, \quad n, m = 0, 1.$$

For $n = m = 0$, we must choose $t = \min(0,0) + 1 = 1$, so that:

$$F_{0,0}(0) = \sum_{j=1}^1 w_j^2 = w_1^2 = 1.$$

For $n = 1, m = 0$ and $n = 0, m = 1$, we must choose $t = \min(1,0) + 1 = 1$, so that:

$$F_{1,0}(0) = F_{0,1}(0) = \sum_{j=1}^1 w_{j-1} w_j = w_0 w_1 = 2.$$

For $n = m = 1$, we must choose $t = \min(1,1) + 1 = 2$, so that:

$$F_{1,1}(0) = \sum_{j=1}^2 j^2 w_{j-1}^2 = w_0^2 + 4w_1^2 = 8.$$

The moment of inertia matrix is the symmetrical matrix

$$F(0) = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$$

with the characteristic equation:

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 8-\lambda \end{vmatrix} = 0,$$

So, it is a polynomial equation of the second degree $\lambda^2 - 9\lambda + 4 = 0$, with

$$\lambda_1 = \frac{9 + \sqrt{65}}{2} \cong 8.53113 \quad \text{and} \quad \lambda_2 = \frac{9 - \sqrt{65}}{2} \cong 0.46887.$$

To the eigenvalue $\lambda_1 = 8.53113$ corresponds the eigenvector $\tilde{a}(0) = (\tilde{a}_0(0), \tilde{a}_1(0)) = (0.25667, 0.9665)$ and to the other eigenvalue $\lambda_2 = 0.46887$

corresponds the eigenvector $\tilde{a}(0) = (\tilde{a}_0(0), \tilde{a}_1(0)) = (0.9665, -0.25667)$, which represent the possible impulse responses of the noise filter.

We choose the eigenvalue $\lambda = 8.53113$, with its corresponding eigenvector $a(0) = \tilde{a}(0) = (0.25667, 0.9665)$ as impulse response of the noise filter.

According to formula (2), the relations that describe the noise filter are:

$$\begin{aligned} (\hat{f}_0, \hat{f}_1, \hat{f}_2) &= a(0) * (x_0, x_1) = (0.25667, 0.9665) * (x_0, x_1) = \\ &= (0.25667x_0, 0.9665x_0 + 0.25668x_1, 0.9665x_1), \end{aligned}$$

in accordance with the discrete convolution algorithm:

$$\begin{array}{rcc} & 0.25667 & 0.9665 \\ & x_0 & x_1 \\ \dots\dots\dots & & \\ 0.25667x_0 & 0.9665x_0 & \\ & 0.25667x_1 & 0.9665x_1 \\ \dots\dots\dots & & \\ 0.51334x_0 & 0.9665x_0 + 0.25667x_1 & 0.9665x_1 \end{array}$$

From (4) we write the resolving kernel:

$$\begin{aligned} (S_0, S_1, S_2) &= (0.25667, 0.9665) * (2, 1) = \\ &= (0.51334, 2.18967, 0.9665), \end{aligned}$$

in accordance with the convolution algorithm

$$\begin{array}{rcc} & 0.25667 & 0.9665 \\ & 2 & 1 \\ \dots\dots\dots & & \\ 0.51334 & 1.933 & \\ & 0.25667 & 0.9665 \\ \dots\dots\dots & & \\ 0.51334 & 2.18967 & 0.9665 \end{array}$$

Hence, the relations between the responses to the impulse f_i by their estimates \hat{f}_i are given by:

$$\begin{aligned} (\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3) &= (S_0, S_1, S_2) * (f_0, f_1, f_2, f_3) = \\ &= (0.51334, 2.18967, 0.9665) * (f_0, f_1, f_2, f_3), \end{aligned}$$

resulting:

$$\begin{aligned} (f_0, f_1, f_2, f_3) &= (S_0, S_1, S_2)^{-1} * (\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3) = \\ (0.51334, 2.18967, 0.9665)^{-1} &* (\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3) = \end{aligned}$$

$$\begin{aligned}
&= (1.948, -8.3093, 31.776) * (\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3) = \\
&= (1.948\hat{f}_0, 1.948\hat{f}_1 - 8.3093\hat{f}_2, 1.948\hat{f}_2 - 8.3093\hat{f}_1 + 31.776\hat{f}_0, \\
&1.948\hat{f}_3 - 8.3093\hat{f}_2 + 31.776\hat{f}_1),
\end{aligned}$$

because $(0,51334, 2.18967, 0.9665)^{-1} = (1.948, -8.3093, 31.776)$, in accordance with the deconvolution algorithm:

$$\begin{array}{cccccc}
1 & 0 & 0 & 0.51334 & 2.18967 & 0.9665 \\
\hline
1 & 4.2655 & 1.8827 & 1.948 & -8.3093 & 31.776 \\
\hline
/ & -4.2655 & -1.8827 & & & \\
& -4.2655 & -18.1946 & & & \\
\hline
& / & 16.3119 & & & \\
& & 16.3119 & & & \\
& & & & & \\
& & & & & /
\end{array}$$

Using formula (1) we obtain:

$$\begin{aligned}
(x_i) &= (f_i) * (w_i) = (f_0, f_1, f_2, f_3) * (w_0, w_1) = \\
&(\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3) * (1.948, -8.3093, 31.776) * (2, 1) = \\
&= (\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3) * (3.896, -14.9706, -55.2427, 31.776) = \\
&= (3.896\hat{f}_0, 3.896\hat{f}_1 - 14.6706\hat{f}_0, 3.896\hat{f}_2 - 14.6706\hat{f}_1 - 55.2427\hat{f}_0, \\
&3.896\hat{f}_3 - 14.6706\hat{f}_2 - 55.2427\hat{f}_1 + 31.776\hat{f}_0).
\end{aligned}$$

At the moment $i = 1$, by using formula (10) we obtain:

$$F_{n,m}(1) = \sum_{j=0}^t (1-j)^2 w_{j-n} w_{j-m}, \quad n, m = 0, 1.$$

For $n = m = 0$ we choose $t = \min(0,0) + 1 = 1$, therefore we have:

$$F_{0,0}(1) = \sum_{j=0}^1 (1-j)^2 w_j^2 = w_0^2 = 4.$$

For $n = 1, m = 0$ and $n = 0, m = 1$, we choose $t = \min(1,0) + 1 = 1$, so that:

$$F_{1,0}(1) = F_{0,1}(1) = \sum_{j=0}^1 (1-j)^2 w_{j-1} w_j = 0.$$

For $n = m = 1$, we choose $t = \min(1,1) + 1 = 2$, therefore:

$$F_{1,1}(\mathbf{1}) = \sum_{j=0}^2 (1-j)^2 w_{j-1}^2 = w_1^2 = 1.$$

The moment of inertia matrix is therefore given by the symmetrical matrix:

$$F(0) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Its characteristic equation is given by:

$$\begin{vmatrix} 4-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0,$$

with the solutions $\lambda_1 = 4$ and $\lambda_2 = 1$.

To the eigenvalue $\lambda_1 = 4$ it corresponds the eigenvector $a'(\mathbf{1}) = (0,1)$ and to the eigenvalue $\lambda_2 = 1$ it corresponds the eigenvector $a''(\mathbf{1}) = (1,0)$, which represent the possible impulse response of the noise filter.

4. Conclusions

1. The example considered is a simple unreal one. We analyze two moments of an earthquake taking place - the initial moment of the earthquake, considered to have a higher intensity and the following moment in which the intensity is reduced by half.
2. Due to the simplicity of our wavelet, we will be able to extract our conclusions only after the first moment of the earthquake.
3. We can confirm our conclusions by applying our deconvolution algorithm and then comparing the data obtained with the ones that we estimated.
4. To deconvolve the discrete data measured, we use a filter, called noise filter, whose impulse response is established based on the following condition: the norm of the second moment of the resolving kernels must be minimal.
5. This minimization is made by using the Lagrange multipliers.
6. Determination of the impulse response of the filter is reduced to a problem of eigenvalues and eigenvectors for a matrix called *the moment of inertia matrix*. The impulse responses of the used filter for deconvolution (inverse problem) are in fact the eigenvectors of this matrix.
7. The main advantage of using the second moment norm is that the moment of inertia matrix is symmetrical and positive defined, which assures that the eigenvalues obtained are real numbers, positive or null.
8. Using discrete convolution and deconvolution, we can obtain a connection between various values that describe the earthquake. Some of those can be measured or predicted, while others cannot. Those who cannot be measured can still be evaluated if the application of the presented theory confirms the anticipation we made.

5. References:

1. Backus G. E., Gilbert J. F., Numerical application of a formation for geophysical inverse problem, *Geophys. J. Roy.Astr.Soc.*,13 (1967) 247-276.
2. Creangă Emil, Munteanu Iulian, *Analiza semnalelor: partea I*, Editura Academica, Galați
3. Lines L. R., Treitel S., Linear inverse theory and deconvolution, *Geophysics*, 47 (1982) 1153-1159.
4. Lines L. R., Treitel S., *Digital filtering with the second moment norm*, *Geophysics*, 48 (1983) 505-514.
5. Oldenburg D., A comprehensive solution to the linear deconvolution problem, *Geophys. J. Roy. Astr. Soc.*, 65 (1981) 331-358.
6. Robinson E. A., Treitel S., *Geophysical signal analysis*, Prentice-Hall. Inc., 1980.
7. Selvi O., A note on digital filtering with the second moment norm, *Geophysics*, 62 (1997) 1315-1320.
8. Ștefănoiu Dan, Matei Ion, Stoica Petre, *Aspecte practice în modelarea și identificarea semnalelor*, UPB
9. Zugravescu Dorel, Mircea Radulian, *The Vrancea seismogenic zone – a natural geodynamics laboratory* , Institutul de geodinamică al Academiei Române, Bucuresti