STATISTICAL LINEARIZATION METHOD OF DUFFING OSCILLATOR UNDER GAUSSIAN WHITE NOISE EXCITATION

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Abstract

In this paper is presented a statistical linearization method of Duffing oscillator under Gaussian white noise excitation. Duffing oscillator's dispersion response to various nonlinearities and spectral intensities is obtained by the statistical linearization method. A comparison is made between dispersions of displacement obtained by this method and Fokker-Planck method in terms of relative errors for different values of nonlinearities and spectral densities.

Keywords: linearization, Duffing oscillator, variance of the response, white noise

Introduction

Statistical linearization methods consist in replacing the nonlinear physical systems with equivalent linear systems, the equivalence being made in relation to certain statistical criteria. The linearization method from deterministic to random vibrations of dynamical systems was extended by Krylov and Bogoliubov [1]. Booton and Caughey developed independently a method of statistical linearization in which the elastic and dissipative characteristics of equivalent linear system are determined from the condition of minimum mean square of the difference between nonlinear and linear equivalent characteristics [2], [3]. In 1963, Caughey extended this method to systems with many degrees of freedom, for which the equations obtained by neglecting the nonlinear terms can be decoupled by the main modes of vibration [4]. In all cases, the necessary maintenance operations within the statistical linearization procedures are carried out in relation to the corresponding probability density corresponding to the equivalent linear system solution, assuming that the excitation is normal distribution and it is completely determined by the averages vector and the covariance matrix, which depends in turn on the equivalent linear system parameters. Equivalence conditions lead to equivalent systems of nonlinear algebraic equations in order to determine these parameters. Although, through the statistical linearization methods, the nonlinear systems replace the linear systems; however, one can obtain certain qualitative results on the effect of nonlinearity types. In order to determine the practical statistical characteristics, other than displacement or velocity relative dispersions - eg absolute acceleration variance – one can go back to the nonlinear equations using only the probability density of displacement and the velocity dispersions or even their variances.

The problem of existence and uniqueness of solutions obtained by statistical linearization it was firstly approached by Spain PD and Iwan W.D. [5]. They showed that the linear system solution is unique. Dimitenberg M.F. [6], Davies H.G. et all [7], [8], Langley R.S. [9] and Fan F.G. and

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G. Ahmadi [10] mentioned that could be multiple solutions derived from statistical linearization. Multiple solutions for linear systems, which are associated with systems subject to random stationary narrow band perturbation are found in [5], [7]. It should be noted that statistical linearization is based on two assumptions: the solutions are Gaussian and systems can be approximated by some linear systems. The first assumption is not satisfied by nonlinear systems in general, while the second implies that the system nonlinearities and disturbance intensities are generally low. As long as two assumptions are approximately fulfilled, multiple solutions might not appear and thus the statistical linearization techniques can be applied.

1. Linearization method

The equivalent linearization method for stationary variance of a Duffing oscillator displacement is applied. The equation of motion is given by [11]:

$$m\ddot{x} + h(x, \dot{x}) = z(t) \tag{1}$$

where $\mathrm{E}[z(t)z(t+\tau)] = 2\pi \mathrm{S}_0 \delta(t)$, S_0 is spectral density of white noise $z(t)$ and

$$h(x, \dot{x}) = c\dot{x} + k(1 + \varepsilon x^2)x \tag{2}$$

where c and k are damping respectively elastic coefficient and ε is the strength of nonlinearity. The system has a linear damping, so it is convenient to assume that the equivalent linear system has the same damping. In this case, the error due to linearization is:

$$\theta(x) = k_e x - \left(kx + k\varepsilon x^3\right) \tag{3}$$

The method involves minimizing the mean square error of $\theta(x)$:

$$\begin{aligned}
\mathrm{E}\left(\theta^2\right) &= \mathrm{E}\left[\left(k_e x - kx - k\varepsilon x^3\right)^2\right] = \\
&= \mathrm{E}\left[k_e^2 x^2 + k^2 x^2 + k^2 \varepsilon^2 x^6 + 2k^2 \varepsilon x^4 - 2k_e k^2 x^2 - 2k_e k^2 \varepsilon x^4\right] = \\
&= k^2 \varepsilon^2 E\left[x^6\right] + 2k\varepsilon\left(k - k_e\right) E\left[x^4\right] + \left(k - k_e\right)^2 E\left[x^2\right].
\end{aligned} \tag{4}$$

Taking the derivative with respect to k_e, it is obtained:

$$\frac{\partial \mathrm{E}[\theta^2]}{\partial k_e} = -2k\varepsilon \mathrm{E}\left[x^4\right] - 2\left(k - k_e\right)\mathrm{E}\left[x^2\right] = 0, \tag{5}$$

and

$$k_e = k\left(1 + \varepsilon \frac{\mathrm{E}\left[x^4\right]}{\mathrm{E}\left[x^2\right]}\right). \tag{6}$$

In order to use this result, it is better to know the second and the fourth moments of the response in each moment of time. Since the excitation is a zero mean Gaussian stochastic process, then the oscillator response is zero mean Gaussian process with equal probability density function:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}}, \tag{7}$$

where the variance σ_x^2 is unknown. It is known that [12] :

$$\mathrm{E}\left[x^2\right] = \sigma_x^2, \quad \mathrm{E}\left[x^4\right] = 3\sigma_x^4 \tag{8}$$

With this result, relation (6) becomes:

$$k_e = k\left(1 + 3\varepsilon\right)\sigma_x^2 \tag{9}$$

Since the excitation is Gaussian white noise with the power spectral density S_0, then the variance of the response of the equivalent linear system is:

$$\sigma_x^2 = \frac{m^2\pi S_0}{ck_e} = \frac{m^2\pi S_0}{ck\left(1 + 3\varepsilon\sigma_x^2\right)}. \tag{10}$$

Solving the equation (9), it results in:

$$\sigma_x^2 = \frac{1}{6\varepsilon}\left(\sqrt{1 + 12\varepsilon\frac{m^2\pi S_0}{ck}} - 1\right). \tag{11}$$

Using the notations:

$$\varsigma = \frac{c}{2\sqrt{km}}, \quad \omega_n^2 = \frac{k}{m} \tag{12}$$

the relation (11) becomes:

$$\sigma_x^2 = \frac{1}{6\varepsilon}\left(\sqrt{1 + 12\varepsilon\sigma_{x_0}^2} - 1\right). \tag{13}$$

where

$$\sigma_{x_0}^2 = \frac{\pi S_0}{2\omega_n^3\varsigma}. \tag{14}$$

From (13), with $S_0 = \varepsilon = 1$, $\varsigma = 0{,}05$, $\omega_n = 1\text{rad/s}$ it obtain $\sigma_x^2 = 3{,}236$.

2. Experimental results

The dispersion of Duffing oscillator displacement obtained by the linearization method in relation to spectral density for different values of nonlinearity and white noise intensity is shown in figure 1. On the other side, figure 2 shows the dispersion of Duffing oscillator displacement obtained by the same method for various nonlinearities and the same spectral density. It can be seen that the displacement variation decreases as the nonlinearity increases and it increases as the spectral density of Gaussian white noise type increases.

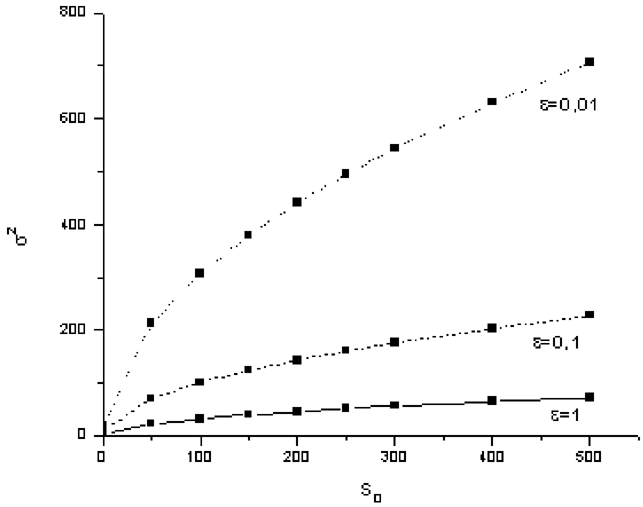

Fig. 1. Duffing oscillator dispersion response to various nonlinearities and spectral intensities obtained by the method of equivalent linearization

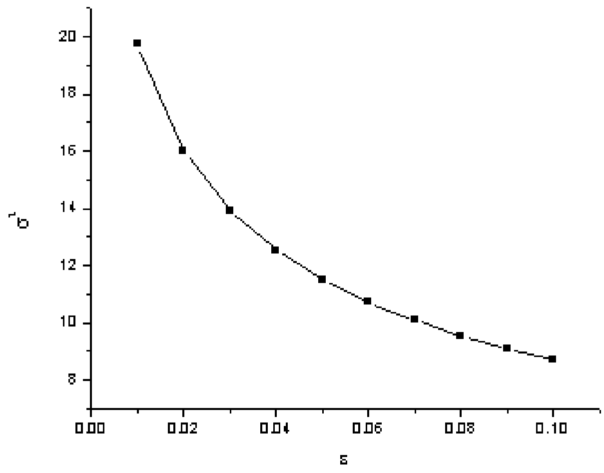

Fig. 2 Duffing oscillator dispersion response to various nonlinearities and the same spectral intensity $S_0 = 1$ obtained by the method of equivalent linearization

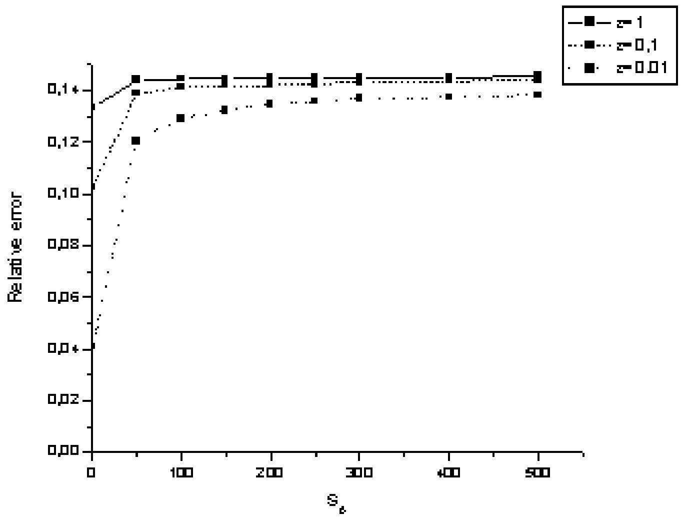

Fig. 3. Relative errors of Duffing oscillator dispersions obtained by Fokker - Planck method and linearization method for various nonlinearities and spectral intensities

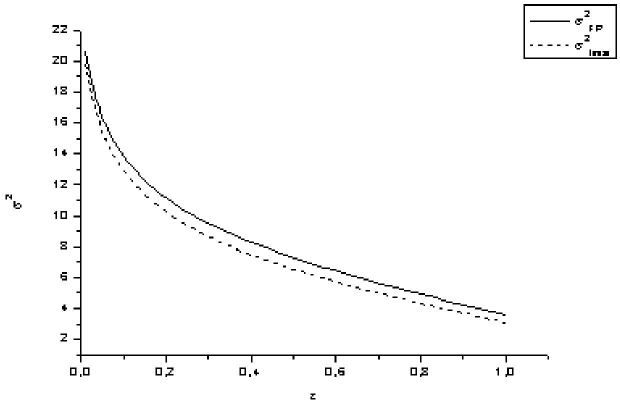

Fig. 4. Dispersion of Duffing oscillator obtained by pr Fokker-Planck Fokker -Planck method and linearization method for various nonlinearities and the same spectral intensity $S_0 = 1$

3. Conclusions

When the nonlinearity increases, the relative errors of the two dispersions obtained by the two methods also increase from 4% when $\varepsilon = 0.01$ to 14% when $\varepsilon = 1$ as you can see in figure 3. As excitation intensity white noise increases, linear equivalence method is inappropriate. A comparison of the variance obtained by Fokker-Planck method and the equivalent linearization method for the Duffing oscillator when $\varsigma = 0.05$, $\omega_n = 1\text{rad/s}$ for various nonlinearities and the same spectral intensity ($S_0 = 1$) are presented in figure 4[13].

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