

LINEAR DISCRETE CONVOLUTION AND ITS INVERSE. PART 2. DECONVOLUTION

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Abstract

We present here several ways for calculating the linear discrete convolution and its inverse - the deconvolution, by direct methods, generator functions, Z-transform, using matrices and MATLAB. These notions was used by author in a series of papers, especially for solve several types of equations.

Keywords: complete and truncated linear discrete convolution and deconvolution.

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Introduction

The convolution is a fundamental concept in mathematics and applications. The use of the convolution and other related notions, as its inverse - the deconvolution, to solve several kinds of equations is particularly of a great importance and was considered by the author in a series of papers, [1] – [6]. They also play a fundamental role in the study of physical linear discrete causal time-invariant systems.

In this part of the paper, we present the reverse notion of linear discrete convolution – the deconvolution, in complete and also in truncated form. We give here the algorithms for calculus of these notions. Are considered both direct algorithms and those based on Matlab, generator functions, Z-transform and matrices. Examples are included.

I. Complete deconvolution

1. Linear discrete convolution and deconvolution in complete form

We call *linear discrete convolution* (or *Cauchy product*) of two finite sequences of real or complex numbers, $a = (a_0, a_1, \dots, a_m)$ and $b = (b_0, b_1, \dots, b_n)$, of the lengths $m + 1$ and $n + 1$, the finite sequence

$$c = a * b = (c_0, c_1, \dots, c_{m+n}) \quad (1)$$

of the length $m + n + 1 = (m + 1) + (n + 1) - 1$, with the terms given by the relations

$$c_0 = a_0 b_0, c_1 = a_1 b_0 + a_0 b_1, \dots, c_k = \sum_{j=0}^k a_{k-j} b_j, \dots, c_{m+n} = a_m b_n. \quad (2)$$

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The convolution is commutative, associative and distributive with respect to the addition of the sequences. The addition and the multiplication with scalars of the sequences are the usual ones.

In the case when the finite sequences a and c are known and $a_0 \neq 0$, we can determine the finite sequence b , so that relation (1) should be satisfied. This sequence is called the *deconvolution* (see [8]) of the sequence c by a , and it is denoted by

$$b = c/a, \tag{3}$$

its terms being given by the relations

$$b_0 = \frac{c_0}{a_0}, b_1 = \frac{1}{a_0}(c_1 - a_1 b_0), \dots, b_k = \frac{1}{a_0} \left(c_k - \sum_{j=0}^{k-1} a_{k-j} b_j \right). \tag{4}$$

Therefore, the deconvolution $b = c/a$ can be calculated by the algorithm

$$\begin{array}{cccc|cccc}
 c_0 & c_1 & \cdots & c_k & \cdots & a_0 & & a_1 & \cdots & a_k & \cdots \\
 c_0 & a_1 b_0 & \cdots & a_k b_0 & \cdots & b_0 = \frac{c_0}{a_0} & b_1 = \frac{c_1 - a_1 b_0}{a_0} & \cdots & & b_k & \cdots
 \end{array}$$

$$\begin{array}{cccc}
 \hline
 / & c_1 - a_1 b_0 & \cdots & c_k - a_k b_0 & \cdots \\
 & c_1 - a_1 b_0 & \cdots & a_k b_1 & \cdots \\
 \hline
 & / & & \cdots &
 \end{array}$$

$$\tag{5}$$

Example. In the examples throughout this first part of the paper, we consider sequences $a = (1, 2, 0, -1, 1)$, $b = (1, 3, -1, -2)$ and their convolution product $c = a * b$. The convolution can be calculated by the algorithm

$$\begin{array}{r}
1 \quad 2 \quad 0 \quad -1 \quad 1 \\
1 \quad 3 \quad -1 \quad -2 \\
\hline
1 \quad 2 \quad 0 \quad -1 \quad 1 \\
\quad 3 \quad 6 \quad 0 \quad -3 \quad 3 \\
\quad \quad -1 \quad -2 \quad 0 \quad 1 \quad -1 \\
\quad \quad \quad -2 \quad -4 \quad 0 \quad 2 \quad -2 \\
\hline
1 \quad 5 \quad 5 \quad -5 \quad -6 \quad 4 \quad 1 \quad -2
\end{array}$$

therefore is $c = a * b = (1, 5, 5, -5, -6, 4, 1, -2)$.

Using the algorithm given above, we now make the deconvolution of the sequence c by a :

$$\begin{array}{r}
1 \quad 5 \quad 5 \quad -5 \quad -6 \quad 4 \quad 1 \quad -2 \\
1 \quad 2 \quad 0 \quad -1 \quad 1 \\
\hline
/ \quad 3 \quad 5 \quad -4 \quad -7 \quad 4 \\
\quad 3 \quad 6 \quad 0 \quad -3 \quad 3 \\
\hline
/ \quad -1 \quad -4 \quad -4 \quad 1 \quad 1 \\
\quad -1 \quad -2 \quad 0 \quad 1 \quad -1 \\
\hline
/ \quad -2 \quad -4 \quad 0 \quad 2 \quad -2 \\
\quad -2 \quad -4 \quad 0 \quad 2 \quad -2 \\
\hline
/
\end{array}
\left| \begin{array}{l}
1 \quad 2 \quad 0 \quad -1 \quad 1 \\
\hline
1 \quad 3 \quad -1 \quad -2
\end{array} \right.$$

Note that by this deconvolution we obtain the sequence b . Analogously, if we deconvolve c by b , it is obtained the sequence a .

2. Computing convolution and deconvolution by generating function

It is called *generating function* (see [10]) of a finite sequence $a = (a_0, a_1, \dots, a_m)$ the polynomial $(G(a))(z) = \sum_{k=0}^m a_k z^k$, $\forall z \in \mathbf{C}$. If $(G(b))(z) = \sum_{k=0}^m b_k z^k$ is the generating function of b , then

$$(G(a * b))(z) = (G(a))(z)(G(b))(z), \quad \forall z \in \mathbf{C}. \quad (6)$$

The convolution product of the sequences a and b can be calculated using the formula (6).

$$\textbf{Example. } (G(c))(z) = (G(a * b))(z) = (G(a))(z)(G(b))(z) = \\ = (1 + 2z - z^3 + z^4)(1 + 3z - z^2 - 2z^3) = 1 + 5z + 5z^2 - 5z^3 - 6z^4 + 4z^5 + z^6 - 2z^7$$

and is obtained the same convolution product $c = a * b = (1, 5, 5, -5, -6, 4, 1, -2)$ as above. This example was given in [7], with a mistake, corrected here.

From formula (6), we can deduce the following formula, useful to calculate the deconvolution of the sequence c by the sequence a :

$$(G(b))(z) = (G(c))(z) / (G(a))(z), \quad \forall z \in \mathbf{C}. \quad (7)$$

$$\textbf{Example. } (G(b))(z) = (G(c))(z) / (G(a))(z) = \\ = (1 + 5z + 5z^2 - 5z^3 - 6z^4 + 4z^5 + z^6 - 2z^7) / (1 + 2z - z^3 + z^4) = 1 + 3z - z^2 - 2z^3.$$

3. Computing convolution and deconvolution by Z-transform

It is called *Z-transform* (see [9]) of a finite sequence $a = (a_0, a_1, \dots, a_m)$, the rational function $(Z(a))(z) = \sum_{k=0}^m a_k z^{-k}$, $\forall z \in \mathbf{C}$. If $(Z(b))(z) = \sum_{k=0}^m b_k z^{-k}$ is Z-transform of b , then

$$(Z(a * b))(z) = (Z(a))(z)(Z(b))(z), \quad \forall z \in \mathbf{C}. \quad (8)$$

The convolution product of the sequences a and b can be calculated using the formula (8).

$$\textbf{Example. } (Z(c))(z) = (Z(a * b))(z) = (Z(a))(z)(Z(b))(z) = \\ = (1 + 2z^{-1} - z^{-3} + z^{-4})(1 + 3z^{-1} - z^{-2} - 2z^{-3}) = \\ = 1 + 5z^{-1} + 5z^{-2} - 5z^{-3} - 6z^{-4} + 4z^{-5} + z^{-6} - 2z^{-7},$$

and is obtained the same convolution product c as above. This example was given in [7], with a mistake, corrected here.

From formula (8), we can deduce the following formula, useful to calculate the deconvolution of the sequence c by the sequence a :

$$(Z(b))(z) = (Z(c))(z) / (Z(a))(z), \quad \forall z \in \mathbf{C}. \quad (9)$$

$$\textbf{Example. } (Z(b))(z) = (Z(c))(z) / (Z(a))(z) =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ -5 & 5 & 5 & 1 & 0 & 0 & 0 & 0 \\ -6 & -5 & 5 & 5 & 1 & 0 & 0 & 0 \\ 4 & -6 & -5 & 5 & 5 & 1 & 0 & 0 \\ 1 & 4 & -6 & -5 & 5 & 5 & 1 & 0 \\ -2 & 1 & 4 & -6 & -5 & 5 & 5 & 1 \end{bmatrix}, \text{ or}$$

$$M(c) = M(a)C(b) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \\ -5 \\ -6 \\ 4 \\ 1 \\ -2 \end{bmatrix},$$

and is obtained the same convolution product $c = a * b$ as above. These examples were given in [7], with some mistakes, corrected here.

From formulas (11), we can deduce the following formulas, useful to calculate the deconvolution of the sequence c by the sequence a :

$$M(b) = M(a)^{-1} M(c), \quad C(b) = M(a)^{-1} C(c). \quad (12)$$

Examples. We have $M(b) = M(a)^{-1} M(c) =$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -7 & 4 & -2 & 1 & 0 & 0 & 0 & 0 \\ 11 & -7 & 4 & -2 & 1 & 0 & 0 & 0 \\ -16 & 11 & -7 & 4 & -2 & 1 & 0 & 0 \\ 21 & -16 & 11 & -7 & 4 & -2 & 1 & 0 \\ -24 & 21 & -16 & 11 & -7 & 4 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ -5 & 5 & 5 & 1 & 0 & 0 & 0 & 0 \\ -6 & -5 & 5 & 5 & 1 & 0 & 0 & 0 \\ 4 & -6 & -5 & 5 & 5 & 1 & 0 & 0 \\ 1 & 4 & -6 & -5 & 5 & 5 & 1 & 0 \\ -2 & 1 & 4 & -6 & -5 & 5 & 5 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 3 & 1 \end{bmatrix}$$

$$C(b) = M(a)^{-1} C(c) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -7 & 4 & -2 & 1 & 0 & 0 & 0 & 0 \\ 11 & -7 & 4 & -2 & 1 & 0 & 0 & 0 \\ -16 & 11 & -7 & 4 & -2 & 1 & 0 & 0 \\ 21 & -16 & 11 & -7 & 4 & -2 & 1 & 0 \\ -24 & 21 & -16 & 11 & -7 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 5 \\ -5 \\ -6 \\ 4 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

therefore, it is obtained the sequence b .

5. MATLAB calculation of complete deconvolution

The deconvolution b of the sequences c by a can be computed in MATLAB by instruction $deconv(c,a)$, that calculate the quotient of the polynomials having terms of sequences c and a as coefficients, hence the quotient of the division of generating functions of these sequences.

Example. By instructions $c = [1\ 5\ 5\ -5\ -6\ 4\ 1\ -2]$, $a = [1\ 2\ 0\ -1\ 1]$, $b = deconv(c,a)$, we obtain $b = [1\ 3\ -1\ -2]$.

II. Truncated deconvolution

6. Linear discrete convolution and deconvolution in truncated form

Very useful is also the truncated form of linear discrete convolution (see [7]), which is calculated with a smaller number of arithmetic operations. Truncated convolution is calculated by the same formula used above, but only for sequences that have the same length, the result also having the same length as the factors. More exactly, let $a = (a_0, a_1, \dots, a_n)$ and $\tilde{b} = (b_0, b_1, \dots, b_n)$ be two sequences of the same length $n+1$. Truncated convolution of a and b is the sequence $\tilde{c} = a * \tilde{b} = (c_0, c_1, \dots, c_n)$ of the same length $n+1$, whose coefficients are given by the formula

$$c_k = \sum_{j=0}^k a_{k-j} b_j, \quad k = 0, 1, \dots, n. \quad (13)$$

This convolution is also commutative, associative, distributive with respect to the addition of the sequences and has the unit $\delta = (1, 0, 0, \dots, 0)$, sequence of length $n+1$.

Truncated convolution can be calculated by each method given above, with fewer calculations. We present below how the truncated convolution can be computed for the sequences from the previous examples, considered now of the same length, by adding a zero to the right of the shortest of them, namely we will obtain $\tilde{c} = (1, 5, 5, -5, -6) = a * \tilde{b}$, for $a = (1, 2, 0, -1, 1)$ and $\tilde{b} = (1, 3, -1, -2, 0)$.

Example. We have the truncated algorithm

$$\begin{array}{r}
1 \quad 2 \quad 0 \quad -1 \quad 1 \\
1 \quad 3 \quad -1 \quad -2 \quad 0 \\
\hline
1 \quad 2 \quad 0 \quad -1 \quad 1 \\
\quad 3 \quad 6 \quad 0 \quad -3 \\
\quad \quad -1 \quad -2 \quad 0 \\
\quad \quad \quad -2 \quad -4 \\
\hline
1 \quad 5 \quad 5 \quad -5 \quad -6
\end{array}$$

hence $\tilde{c} = a * \tilde{b} = (1, 5, 5, -5, -6)$.

Truncated deconvolution is calculated by the same procedures as in complete case, except that now all the sequences involved in the calculation, including partial results, have the same length.

Example. The truncated deconvolution $\tilde{c}/a = \tilde{b}$ can be computed by the algorithm

$$\begin{array}{r}
1 \quad 5 \quad 5 \quad -5 \quad -6 \\
1 \quad 2 \quad 0 \quad -1 \quad 1 \\
\hline
/ \quad 3 \quad 5 \quad -4 \quad -7 \\
\quad 3 \quad 6 \quad 0 \quad -3 \\
\hline
\quad / \quad -1 \quad -4 \quad -4 \\
\quad \quad -1 \quad -2 \quad 0 \\
\hline
\quad \quad / \quad -2 \quad -4 \\
\quad \quad \quad -2 \quad -4 \\
\hline
\quad \quad \quad /
\end{array}$$

7. Computing truncated deconvolution by generator function

Example. We consider only polynomials up to degree 4, so

$$\begin{aligned}
(G(b))(z) &= (G(c))(z) / (G(a))(z) = \\
&= (1 + 5z + 5z^2 - 5z^3 - 6z^4) / (1 + 2z - z^3 + z^4) = 1 + 3z - z^2 - 2z^3,
\end{aligned}$$

and is obtained the same deconvolution $\tilde{c}/a = \tilde{b}$ as above.

8. Computing truncated convolution by Z-transform

Example. We consider only Z-transform up to power -4 , so

$$\begin{aligned} (Z(b))(z) &= (Z(c))(z) / (Z(a))(z) = \\ &= (1 + 5z^{-1} + 5z^{-2} - 5z^{-3} - 6z^{-4}) / (1 + 2z^{-1} - z^{-3} + z^{-4}) = 1 + 3z^{-1} - z^{-2} - 2z^{-3} \end{aligned}$$

and is obtained the same deconvolution $\tilde{c}/a = \tilde{b}$ as above.

9. Matrix calculation of convolution and deconvolution in truncated form

In matrix method, for the sequence $a = (a_0, a_1, \dots, a_n)$, we consider the matrices $M(a) \in M_{n+1, n+1}$ and $C(a) \in M_{n+1, 1}$ given by relations

$$M(a) = \begin{bmatrix} a_0 & 0 & \cdots & 0 & 0 \\ a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 & 0 \\ a_n & a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix}, \quad C(a) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix}. \quad (14)$$

Then, the convolution product $\tilde{c} = (c_0, c_1, \dots, c_n) = a * \tilde{b}$ of a and $\tilde{b} = (b_0, b_1, \dots, b_n)$ is calculated by relations

$$M(a * \tilde{b}) = M(a)M(\tilde{b}), \quad C(a * \tilde{b}) = M(a)C(\tilde{b}), \quad (15)$$

and the deconvolution $\tilde{c}/a = \tilde{b}$ by relations

$$M(\tilde{b}) = M(a)^{-1}M(\tilde{c}), \quad C(\tilde{b}) = M(a)^{-1}C(\tilde{c}). \quad (16)$$

Example. We have

$$M(\tilde{b}) = M(a)^{-1}M(\tilde{c}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 \\ -7 & 4 & -2 & 1 & 0 \\ 11 & -7 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 \\ 5 & 5 & 1 & 0 & 0 \\ -5 & 5 & 5 & 1 & 0 \\ -6 & -5 & 5 & 5 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ -1 & 3 & 1 & 0 & 0 \\ -2 & -1 & 3 & 1 & 0 \\ 0 & -2 & -1 & 3 & 1 \end{bmatrix}, \text{ or}$$

$$C(\tilde{b}) = M(a)^{-1}C(\tilde{c}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 \\ -7 & 4 & -2 & 1 & 0 \\ 11 & -7 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 5 \\ -5 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ -2 \\ 0 \end{bmatrix},$$

and is obtained the same truncated deconvolution $\tilde{c}/a = \tilde{b}$ as above.

10. MATLAB calculation of truncated convolution

We can easily create in MATLAB a new instruction, named $\tilde{b} = tdeconv(\tilde{c}, a)$, to compute the truncated deconvolution $\tilde{b} = \tilde{c}/a$, with a, \tilde{b}, \tilde{c} of same length.

Example. By instructions $\tilde{c} = [1\ 5\ 5\ -5\ -6]$, $a = [1\ 2\ 0\ -11]$, $\tilde{b} = tdeconv(\tilde{c}, a)$, we obtain $\tilde{b} = [1\ 3\ -1\ -2\ 0]$.

11. The convolution inverse and its application

Denoting the inverse of the finite sequence a by $a^{-1} = \delta/a$, where $\delta = (1, 0, 0, \dots, 0)$ is of the same length as a, \tilde{b} and \tilde{c} , we have

$$\tilde{b} = \tilde{c}/a = \tilde{c} * a^{-1}. \quad (17)$$

Therefore, the deconvolution $\tilde{b} = \tilde{c}/a$ can be calculated by the deconvolution $a^{-1} = \delta/a$, followed by convolution product (17). For a single deconvolution, this procedure is not too useful. But there are situations when must to be calculated deconvolutions of several different sequences by the same sequence. Then, the mentioned procedure is very useful.

Example. The convolutional inverse of a is given by the algorithm

$$\begin{array}{r}
\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & -1 & 1 \\
\hline
-2 & 0 & 1 & -1 & \\
-2 & -4 & 0 & 2 & \\
\hline
& / & 4 & 1 & -3 \\
& & 4 & 8 & 0 \\
& & \hline
& & / & -7 & -3 \\
& & & -7 & -14 \\
& & & \hline
& & & / & 11 \\
& & & & 11 \\
& & & & \hline
& & & & /
\end{array}
\end{array}
\quad
\begin{array}{ccccc}
1 & 2 & 0 & -1 & 1 \\
\hline
1 & -2 & 4 & -7 & 11
\end{array}$$

hence $a^{-1} = (1, -2, 4, -7, 11)$. Therefore, we have

$$\tilde{c}/a = \tilde{c} * a^{-1} = (1, 5, 5, -5, -6) * (1, -2, 4, -7, 11) = (1, 3, -1, -2, 0) = \tilde{b}.$$

III. Linear discrete deconvolution of infinite (unilateral) sequences

For two infinite (unilateral) sequences $a = (a_k : k \in \mathbf{N})$ and $b = (b_k : k \in \mathbf{N})$, the linear convolution $c = a * b = (c_k : k \in \mathbf{N})$ is defined by formula (13) for every $k \in \mathbf{N}$. This notion corresponds to multiplication of the formal series associated as generator functions of the considered sequences. In turn, the deconvolution will be correspond to division of formal series. The calculation methods for truncated convolution and deconvolution presented here and in [7] can be applied also in the infinite case, if we consider an arbitrary index k .

Examples. If $p, q \neq 0$ are complex numbers, then

$$(p^n) * (q^n) = \left(\sum_{k=0}^n p^k q^{n-k} \right) = \left(q^n \sum_{k=0}^n \left(\frac{p}{q} \right)^k \right) = \left(q^n \frac{1 - p^{n+1}/q^{n+1}}{1 - p/q} \right) = \left(\frac{q^{n+1} - p^{n+1}}{q - p} \right),$$

if $p \neq q$, $(q^n) * (q^n) = \left(\sum_{k=0}^n q^n \right) = ((n+1)q^n)$, $(1, -1, 0, 0, \dots)^{-1} = (1, 1, \dots)$.

IV. Convolution radicals

If the sequence $a = (a_0, a_1, a_2, \dots)$, with $a_0 \neq 0$, is given, then the *convolution radical of second order* $\sqrt{a} = x = (x_0, x_1, x_2, \dots)$, can be determined by relation $x * x = a$, considered as truncated convolution in finite sequences case, that gives the system

$x_0^2 = a_0$, $2x_0x_1 = a_1$, $2x_0x_2 + x_1^2 = a_2$, $2x_0x_3 + 2x_1x_2 = a_3$, $2x_0x_4 + 2x_1x_3 + x_2^2 = a_4$
and so,

with solutions $x_0 = \pm\sqrt{a_0}$, $x_1 = \pm\frac{a_1}{2\sqrt{a_0}}$, $x_2 = \pm\frac{4a_0a_2 - a_1^2}{8\sqrt{a_0}}$,

$x_3 = \pm\frac{8a_0a_3 - 4a_0a_1a_2 + a_1^3}{16a_0\sqrt{a_0}}$, and so.

Examples. $\sqrt{(1,4,9,16,25,36)} = \pm\left(1, 2, \frac{5}{2}, 3, \frac{27}{8}, \frac{15}{4}\right)$,

$\sqrt{(1,2,3,4,5,\dots)} = \pm(1,1,\dots)$.

References

- [1] M. Cîrnu, Solving difference and differential equations by discrete deconvolution, *UPB Scientific Bulletin*, A, **69** (2007) no. 1, 13-26.
- [2] M. Cîrnu, Determination of particular solutions of nonhomogeneous linear differential equations by discrete deconvolution, *UPB Scientific Bulletin*, A, **69** (2007) No. 2, 3-16.
- [3]. M. Cîrnu, Approximate calculus by deconvolution of the polynomial roots, *UPB Scientific Bulletin*, A **69** (2007) No. 4, 9-22.
- [4] M. Cîrnu, "First order differential recurrence equations with discrete auto-convolution", *International Journal of Mathematics and Computation*, **4** (2009) S09, 124-128.
- [5] M. I. Cîrnu, F. D. Frumosu, Initial value problems for nonlinear differential equations solved by differential transform method, I, *Journal of Information Systems and Operations Management*, Vol. 3, No. 1, July 2009, 102-107.
- [6] M. I. Cîrnu, F. D. Frumosu, Initial value problems for nonlinear differential equations solved by differential transform method, II, *Journal of Information Systems and Operations Management*, Vol. 3, No. 2, December 2009, 381-387.
- [7] M. I. Cîrnu, Linear discrete convolution and its inverse. Part 1. Convolution, *Journal of Information Systems and Operations Management*, Vol. 4, No. 1, May 2010, 129-137
- [8] M. A. Cuenod, Introduction a l'analyse impulsionelle. Principe et application, *Dunod, Paris*, 1970.
- [9] E. I. Jury, Theory and application of the Z-transform method, Krieger Pub Co, 1973.
- [10] H. S. Wilf, Generating functionology, *Academic Press*, 1990.