

**INITIAL VALUE PROBLEMS FOR NONLINEAR
DIFFERENTIAL EQUATIONS SOLVED BY
DIFFERENTIAL TRANSFORM METHOD. II**

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ABSTRACT

In the paper [2] we presented how can solve initial value problems for non-linear differential equations using the differential transform. We did this by solving in a rigorous way, based on mathematically induction, an example taken from the paper [3]. Here present the same for another example taken from the paper [3]. In [3] and also in [6] and respective [4], these examples was solved as boundary value problems. The chosen example is proved to have an exact solution. A Maple program is also included.

Keywords: Nonlinear differential equations, initial value problems, differential transform method.

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DIFFERENTIAL TRANSFORM METHOD

The Mac Laurin differential transform F of an indefinite differential function $u(x)$ is defined as being the following numerical sequence:

$$F(u(x)) = \left(\frac{1}{k!} u^{(k)}(0) \right) = (U(k)).$$

The inverse differential transform F^{-1} associate to the numerical sequence $(U(k))$ the function $u(x)$ defined as follows:

$$u(x) = \sum_{k=0}^{\infty} U(k)x^k, \text{ with } U(k) = \frac{1}{k!} u^{(k)}(0).$$

$$\text{So } F^{-1}((U(k))) = \sum_{k=0}^{\infty} U(k)x^k.$$

If $u(x) = \sum_{k=0}^{\infty} U(k)x^k$ and $v(x) = \sum_{k=0}^{\infty} V(k)x^k$, we have the following properties:

$$F(u(x) + v(x)) = (U(k) + V(k)),$$

$$F(au(x)) = (aU(k)),$$

$$F(u^{(m)}(x)) = \left(\frac{(m+k)!}{k!} U(m+k) \right),$$

particularly,

$$F(u'(x)) = ((k+1)U(k+1)),$$

$$F(u''(x)) = ((k+1)(k+2)U(k+2)),$$

$$F(u(x)v(x)) = (U(k)) * (V(k)) = \left(\sum_{k=0}^n U(k)V(n-k) \right)$$

$$\text{and } F(x^n) = (\delta(k-n))$$

$$\text{where } \delta(k-n) = \begin{cases} 1 & , \quad k = n \\ 0 & , \quad k \neq n \end{cases}.$$

EXAMPLE

We consider the initial value problem composed from the nonlinear and non-homogeneous differential equation:

$$u''(x) + u^2(x) = x^4 + 2, \quad 0 < x < 1, \\ \text{and the following initial values: } u(0) = 0, u'(0) = 0.$$

Using the method presented above, it results the following recurrence relation:

$$U(k+2) = \frac{1}{(k+1)(k+2)} \left[- \sum_{l=0}^k U(l)U(k-l) + \delta(k-4) + 2\delta(k) \right],$$

$$k = 0, 1, 2, \dots,$$

with the following transformed initial values:

$$u(0) = U(0) = 0, U(1) = u'(0) = a.$$

In the above recurrence relation, k takes the values 0,1,2,3,4,5. We obtain

$$U(2) = \frac{1}{2} [-U(0)^2 + 2\delta(0)] = \frac{1}{2} \cdot 2 = 1,$$

$$U(3) = -\frac{1}{3} U(0)U(1) = 0,$$

$$U(4) = -\frac{1}{12} [2U(0)U(2) + U(1)^2] = -\frac{a^2}{12},$$

$$U(5) = -\frac{1}{10} [U(0)U(3) + U(1)U(2)] = -\frac{a}{10},$$

$$U(6) = \frac{1}{30} [-2U(0)U(4) - 2U(1)U(3) - U(2)^2 + \delta(0)] = \frac{1}{30} [-1 + 1] = 0,$$

$$U(7) = -\frac{1}{21} [U(0)U(5) + U(1)U(4) + U(2)U(3)] = \frac{a^2}{252}.$$

But $a=0$, so we have $U(0) = U(1) = U(3) = U(4) = U(5) = U(6) = U(7) = 0$ and $U(2) = 1$. For $n \geq 6$, we suppose that $U(k) = 0$, $k=3,4,\dots,n+1$. From the above recurrence relation we have

$$U(n+2) = -\frac{1}{(n+1)(n+2)} \left[2U(0)U(n) + 2U(1)U(n-1) + 2U(2)U(n-2) + \sum_{l=3}^{n-3} U(l)U(n-l) \right] = 0.$$

In conformity with induction axiom, we have $U(n) = 0$, $n \geq 3$, therefore the exact solution of the initial value problem is:

$$u(x) = x^2.$$

MAPLE PROGRAM

We use the above given recurrence relation to determine the first coefficients of power series development of the solution.

> restart;

The equation is :

> Diff(u(x),x\$2)+u(x)^2-x^4-2=0;

$$\left(\frac{d^2}{dx^2} u(x) \right) + u(x)^2 - x^4 - 2 = 0$$

The initial values are :

> u(0)=0;

> Diff(u(0),x)=0;

$$u(0) = 0$$

$$\frac{d}{dx} u(0) = 0$$

The transformed initial values :

> U(0):=0;

> U(1):=1;

$$U(0) := 0$$

$$U(1) := a$$

The recurrence relation is :

$$> U(k+2) := 1/((k+1)*(k+2))*(-\text{Sum}(U(l)*U(k-l), l=0..k) + \text{delta_procedura}(k,4) + 2*\text{delta_procedura}(k,0));$$

The values of U(2), U(3), U(4), U(5) ... U(11) are :

> for k from 0 to 9 do

$$U(k+2) := 1/((k+1)*(k+2))*(-\text{sum}(U(l)*U(k-l), l=0..k) + \text{delta_procedura}(k,4) + 2*\text{delta_procedura}(k,0));$$

end do;

$$U(2) := 1$$

$$U(3) := 0$$

$$U(4) := -\frac{a^2}{12}$$

$$U(5) := -\frac{a}{10}$$

$$U(6) := 0$$

$$U(7) := \frac{a^3}{252}$$

$$U(8) := \frac{11 a^2}{1680}$$

$$U(9) := \frac{a}{360}$$

$$U(10) := -\frac{a^4}{6048}$$

$$U(11) := -\frac{19 a^3}{55440}$$

The symbolic u(x) is :

$$> ux := \text{Sum}(U(i)*x^i, i=0..4);$$

$$ux := \sum_{i=0}^4 U(i) x^i$$

$$> ux := \text{sum}(U(i)*x^i, i=0..9);$$

$$ux := a x + x^2 - \frac{1}{12} a^2 x^4 - \frac{1}{10} a x^5 + \frac{1}{252} a^3 x^7 + \frac{11}{1680} a^2 x^8 + \frac{1}{360} a x^9$$

The exact solution of u(x) in a=0, is:

```
> a:=0;  
a := 0
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```
> ux;  
 $x^2$ 
```

The graphic of the exact solution is:

```
> plot(ux, x = 0..1);
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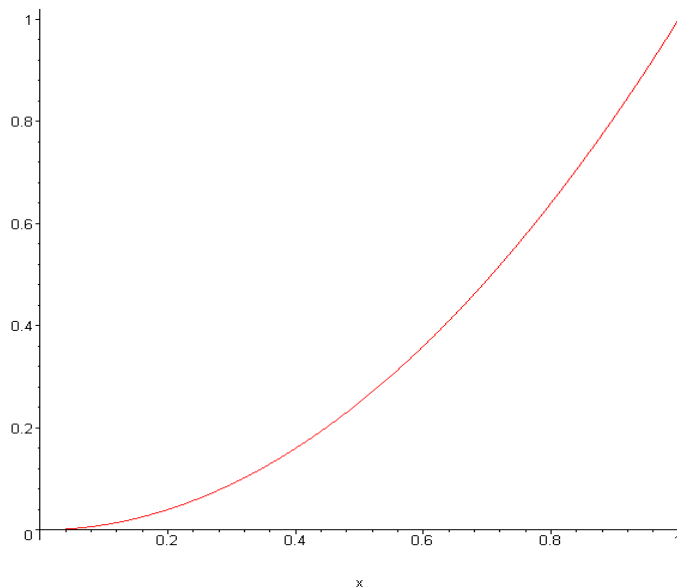


Figure: The graph of the exact solution $u(x) = x^2$.

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