

SYMMETRICALLY-CLOSED VALUATIONS ON $K(X_1, \dots, X_n)$ Cătălina Vişan^{1*}**ABSTRACT:**

In the domain of valued fields K and, more specifically, the extensions of valuations on K to $K(X_1, \dots, X_n)$, an important category of extensions, namely the symmetric extensions with respect to the indeterminates $X_1, \dots, X_n$, came recently into attention. This paper deals with the characterization of the symmetric extensions of valuations on K to $K(X_1, \dots, X_n)$ that are not symmetrically-open, thus not having a simple closed-form expression for definition. We will define the notions of *symmetry degree* and the *most relevant restriction* (symmetrically-wise) of an extension of a valuation from K to $K(X_1, \dots, X_n)$ and we will use these quantities to characterize this unfriendly class of symmetrically-closed extensions.

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1. INTRODUCTION

The theory of valuation is a rather young field of research in mathematics, becoming a topic of interest only in the last century, when important mathematicians started to contribute important books and papers to this domain ([1], [2], [3], [4]).

The classification of the extensions of a valuation, from K to $K(X_1, \dots, X_n)$ is a difficult open problem in algebra, leading to serious issues in the domain of algebraic geometry once we get with the analysis to the second indeterminate (X_2) and, thus facing the algebraic closure of the field $K(X_1)$. Several directions in the research of simplifications of this problem were developed, based on the solid foundation laid in [5], [6], [7], [8] and [9], by the complete characterization of the extensions from K to $K(X)$, in [10], [11], [12] and [13].

One of such direction was proposed in [14], by defining a special class of extensions of a valuation from K to $K(X_1, \dots, X_n)$, called *symmetric valuations*, which treats in an undifferentiated way the n indeterminates. By analyzing this simpler class of extensions and by reducing an extension in general case to a symmetrical one, the mentioned difficult problem is avoided. The first major result in this direction was communicated in [15], by offering a complete classification of those symmetric extensions, called *symmetrically-*

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open extensions, that may prolong their symmetry indeterminately, to any number of extra indeterminates.

This paper intends to analyze the remaining extensions, the *symmetrically-closed* ones, that have a limitation in the number of indeterminates for which their symmetry may still be preserved.

2. GENERAL NOTATIONS AND DEFINITIONS

Let's consider a field K and v a valuation on K . We denote this relation by the pair (K, v) . We denote by k_v the *residue field*, by G_v the *value group*, by O_v the *valuation ring* and by M_v the *maximal ideal* of v . We denote by $\rho_v: O_v \rightarrow k_v$ the *residual homeomorphism*. For $x \in O_v$ we denote by $x^* = \rho_v(x)$ its image in k_v .

Given two valuations, u and u' , on K , we say that u is equivalent to u' and write $u \cong u'$, if there exists an isomorphism of order groups $j: G_u \rightarrow G_{u'}$ such that $u' = ju$.

Let K'/K be an extension of fields. We call a valuation v' on K' an *extension* of v if $v'(x) = v(x)$ for all x in K . If v' is an extension of v we may canonically identify $k_{v'}$ with a subfield of k_v and $G_{v'}$ with a subgroup of G_v .

Let (K, v) be a valued field. By choosing $\bar{K}$ an algebraic closure of K and $\bar{v}$ an extension of v to $\bar{K}$, the residual field of $\bar{v}$ is, in fact, an algebraic closure of k_v and the value group of $\bar{v}$ will be $\mathbf{Q}G_v$, namely the smallest divisible group that still contains G_v .

We denote by $K(X)$ the field of rational fractions of an indeterminate X over K and with $K[X]$ the ring of polynomials of an indeterminate X over K .

Let u be an extension of v to $K(X)$. We say that u is a *residual-transcendental extension* (*r.t.-extension*) if k_u / k_v is a transcendental extension of fields. If not, but we still have $G_u \subseteq \mathbf{Q}G_v$, we say that u is a *residual-algebraic torsion extension* (*r.a.t.-extension*) and when $G_u \not\subseteq \mathbf{Q}G_v$, we say that u is a *residual-algebraic free extension* (*r.a.f.-extension*). More details about this classification may be found in [9].

In paper [14] a *symmetric valuation* (with respect to $X_1, \dots, X_n$) was defined as a valuation w on $K(X_1, \dots, X_n)$, $n \geq 2$, such that, given any permutation π of $\{1, 2, \dots, n\}$ and any $f \in K(X_1, \dots, X_n)$, we have

$$w(f(X_1, X_2, \dots, X_n)) = w(f(X_{\pi(1)}, X_{\pi(2)}, \dots, X_{\pi(n)})).$$

In this case we denote by $\pi f(X_1, X_2, \dots, X_n) = f(X_{\pi(1)}, X_{\pi(2)}, \dots, X_{\pi(n)})$, the automorphism $f \rightarrow \pi f$ of $K(X_1, \dots, X_n)$ that leaves the symmetric fractions of polynomials in $K(X_1, \dots, X_n)$ unchanged.

Let w be a symmetric valuation on $K(X_1, \dots, X_n)$. Let $\overline{K(X_1, \dots, X_n)}$ be an algebraic closure of $K(X_1, \dots, X_n)$ and $\overline{w}$ an extension of w from $K(X_1, \dots, X_n)$ to $\overline{K(X_1, \dots, X_n)}$.

We say that $\overline{w}$ *extends the symmetry* of w if, for any partition of $\{1, 2, \dots, n\} = \{i_1, i_2, \dots, i_m\} \cup \{j_1, j_2, \dots, j_{n-m}\}$, with $0 \leq m < n$, the restriction of $\overline{w}$ to $\overline{K(X_{i_1}, \dots, X_{i_m})(X_{j_1}, \dots, X_{j_{n-m}})}$ is symmetric with respect to $X_{j_1}, \dots, X_{j_{n-m}}$, where $\overline{K(X_{i_1}, \dots, X_{i_m})}$ is the closure of $K(X_{i_1}, \dots, X_{i_m})$ in $\overline{K(X_1, \dots, X_n)}$. For such an extension we denote by:

$$\delta_a := \overline{w}(X - a), \text{ for any } a \in \overline{K}, \text{ where } X \text{ is arbitrarily chosen from } X_1, \dots, X_n;$$

$$M_{\overline{w}} := \{\delta_a / a \in \overline{K}\};$$

and for any i , such that $0 \leq i \leq n$, we denote by:

$$K_i := K(X_1, \dots, X_i), \text{ with the convention } K_0 = K;$$

$$u_i := \text{the restriction of } w \text{ to } K_i, \text{ with the conventions } u_0 = v, u_n = w;$$

$$O_i, G_i, \text{ resp. } k_i := \text{the valuation ring, valuation group, resp. residual field of } u_i;$$

$$M_i := \{\overline{w}(X_i - \rho) / \rho \in \overline{K(X_1, \dots, X_{i-1})}\}, \text{ for } i \geq 1.$$

We call the *freedom degree* of the extension w (with respect to v) the quantity

$$\text{freedeg } w = \text{card } \{i \in \{1, \dots, n\} / G_i \cap \mathbf{Q}G_{i-1} \neq G_i\}.$$

and we notice, due to [9], that $\text{freedeg } w$ represents the number of intermediate extensions from v on K to w on $K(X_1, \dots, X_n)$ that are residual-algebraic free and this number is independent on the order the indeterminates $X_1, \dots, X_n$ are taken into account.

An extension w , of a valuation v from K to $K(X_1, \dots, X_n)$, symmetric with respect to $X_1, \dots, X_n$, is called *symmetrically-open* (with respect to $X_1, \dots, X_n$) if, adding any number of other indeterminates (elements transcendental and algebraically independent over $K(X_1, \dots, X_n)$), $X_{n+1}, \dots, X_{n+r}$, there exists a symmetric extension of it to $K(X_1, \dots, X_{n+r})$ with respect to $X_1, \dots, X_{n+r}$.

3. CHARACTERIZATION OF THE SYMMETRICALLY-OPEN VALUATIONS

In [15; §4, §5] a couple of results were obtained regarding the symmetrically-open extensions, which we will summarize in this chapter.

Observation 3.1: If w is symmetrically-open with respect to $X_1, \dots, X_n$, with $n \geq 2$, then it is symmetrically-open with respect to $X_1, \dots, X_i$, for $i < n$. Dually, for any symmetrically-open extension with respect to $X_1, \dots, X_n$ there exists an extension of it, symmetrically-open with respect to $X_1, \dots, X_i$, for all $i > n$, with $\text{tr.deg}(K(X_1, \dots, X_i) : K) = i$.

Observation 3.2: A Gaussian extension is symmetrically-open. This means that, if we formally extend the definition above for $n = 0$, we can say that any extension is (trivially) symmetrically-open with respect to the void set.

Observation 3.3: For a chain of symmetrically-open extensions, built using Observation 3.1, there exists a chain of extensions to the algebraic closures (of the fields each of the extensions in the original chain are defined on), such that their symmetry is also extended.

Observation 3.4: A symmetric extension is symmetrically-open if and only if it may be extended to a symmetric valuation on $K(X_1, \dots, X_{n+1})$ that has an extension further to $\overline{K(X_1, \dots, X_{n+1})}$ which extends its symmetry.

Observation 3.5: If w is symmetrically-open with respect to $X_1, \dots, X_n$ then:

$$0 \leq \text{freedeg } w \leq 2;$$

$$n - 2 \leq \text{tr.deg}(k_w : k_v) \leq n;$$

$$n - 1 \leq \text{freedeg } w + \text{tr.deg}(k_w : k_v) \leq n.$$

The complete characterization of the symmetrically-open extensions is given through the following result.

Theorem 3.6 [15; Theorem 5.1]: Let w be a symmetrically-open extension of a valuation v , from K to $K(X_1, \dots, X_n)$, with $n \geq 2$, a fixed algebraic closure $\overline{K(X_1, \dots, X_n)}$ and $\bar{w}$ that extends the symmetry of w to $\overline{K(X_1, \dots, X_n)}$. Then w may be in one of the following possible situations:

(I) $\text{freedeg } w + \text{tr.deg } (k_w : k_v) = n$ and, in this case, w is defined by a triplet (a, δ, ε) , in which we have $a \in \overline{K}$, $\delta \in \mathbb{Z} \times \mathbf{Q}G_v$ and $\varepsilon \in \mathbb{Z} \times \mathbb{Z} \times \mathbf{Q}G_v$, $\varepsilon > \delta$ such that, for any $F \in K[X_1, \dots, X_n]$ written as:

$$F = \sum_{(i_1, \dots, i_n) \in I} f_{i_1, \dots, i_n} \cdot g^{i_1}(X_2 - X_1)^{i_2} \cdots (X_n - X_1)^{i_n},$$

$$\text{with } f_{i_1, \dots, i_n} \in K[X_1], \deg f_{i_1, \dots, i_n} < \deg g$$

where I is a finite set of n -uples of indices and $g \in K[X_1]$ is the minimal monic polynomial of a over K , we get:

$$w(F) = \inf_{(i_1, \dots, i_n) \in I} \left(\bar{v}(f_{i_1, \dots, i_n}(a)) + i_1 \cdot \gamma + (i_2 + \dots + i_n) \cdot \varepsilon \right),$$

$$\text{with } \gamma = \sum_{a' \in \overline{K}, g(a')=0} \inf \left(\delta_a, \bar{v}(a' - a) \right)$$

(II) $\text{freedeg } w + \text{tr.deg } (k_w : k_v) = n - 1$ and, in this case, w is the limit of an ordered system of extensions of type (I), that have in their definition the same value for ε .

4. SYMMETRICALLY-CLOSED VALUATIONS

We may now move to the *symmetrically-closed* valuations, which are simply those that are symmetric, but not symmetrically-open. We will start by defining a quantity that describes the extent to which an ordinary valuation may be restricted or extended to a symmetric valuation.

Definition 4.1: Let w be an extension of a valuation v on K to $K(X_1, \dots, X_n)$. We will denote by $\text{symmdeg } w$ and call the *symmetry degree* of w (with respect to $X_1, \dots, X_n$) the following quantity, case-dependently:

(I) when w is not symmetric with respect to $X_1, \dots, X_n$, $\text{symmdeg } w$ is the largest k , with $1 \leq k < n$, such that, for any set $\{X_{i_1}, \dots, X_{i_k}\} \subset \{X_1, \dots, X_n\}$, we are guaranteed that $w|_{K(X_{i_1}, \dots, X_{i_k})}$ is symmetric with respect to $X_{i_1}, \dots, X_{i_k}$;

(II) when w is symmetric with respect to $X_1, \dots, X_n$, $\text{symmdeg } w$ is either the largest k , with $k \geq n$, such that, for any $X_{n+1}, \dots, X_k$ transcendental and algebraically independent

over $K(X_1, \dots, X_n)$, there exists ω an extension of w to $K(X_1, \dots, X_k)$ which is symmetric with respect to $X_1, \dots, X_k$ or ∞ (infinity), if such largest k doesn't exist.

There are a couple of observations immediately following the definition:

Observation 4.2: $1 \leq \text{symmdeg } w \leq \infty$;

Observation 4.3: w is symmetric with respect to $X_1, \dots, X_n$ if and only if $\text{symmdeg } w \geq n$; to be noted that the statement holds also for the trivial case $n = 0$, in which case, given the fact that we may create an arbitrarily long Gaussian chain of extensions further on, we have $\text{symmdeg } w = \infty$.

Observation 4.4: w is symmetrically-open if and only if $\text{symmdeg } w = \infty$.

We are now interested in comparing the symmetry degrees of two valuations, out of which one is the restriction (or isomorphic with the restriction) of the other one. We will find out that, by extending a valuation to more indeterminates (mutually algebraically independent) this degree decreases or, at most, remain unchanged and this holds both for symmetric and asymmetric valuations.

Proposition 4.5: Let w be an extension of v on K to $K(X_1, \dots, X_n)$ and w' an extension of it to $K(X_1, \dots, X_{n+r})$, where $X_{n+1}, \dots, X_{n+r}$ are transcendental and algebraically independent over $K(X_1, \dots, X_n)$. In this case we get:

$$\text{symmdeg } w \geq \text{symmdeg } w'$$

PROOF:

Case **(I)**: Suppose, first, that $\text{symmdeg } w < n$. This means that $\text{symmdeg } w = k$, where k is the largest integer, in range $1 \leq k < n$, such that, for any $\{X_{i_1}, \dots, X_{i_k}\} \subset \{X_1, \dots, X_n\}$ gives that $w|_{K(X_{i_1}, \dots, X_{i_k})}$ is symmetric with respect to $X_{i_1}, \dots, X_{i_k}$.

Let's assume, by *reductio ad absurdum*, that $\text{symmdeg } w' = l > k$. Then, for any $\{X_{j_1}, \dots, X_{j_l}\} \subset \{X_1, \dots, X_{n+r}\}$ we get that $w'|_{K(X_{j_1}, \dots, X_{j_l})}$ is symmetric with respect to $X_{j_1}, \dots, X_{j_l}$ and let $k' = \min(n, l)$. Let now $\{X_{i_1}, \dots, X_{i_{k'}}\} \subset \{X_1, \dots, X_n\}$, but, since $\{X_{j_1}, \dots, X_{j_{k'}}\}$ is a subset also of $\{X_1, \dots, X_{n+r}\}$, it follows that w' is symmetric with

respect to $X_{i_1}, \dots, X_{i_k}$, which means that so it is w , and this holds for any $\{X_{i_1}, \dots, X_{i_k}\} \subset \{X_1, \dots, X_n\}$, obviously leading to $k \geq k'$. We have now got a contradiction, as $k < n$ and $k < l$.

Case (II): Let's move on to the second case, where $n \leq \text{symmdeg } w < \infty$. According to definition we have $\text{symmdeg } w = k$, where k is the largest integer, with $k \geq n$, such that, for any $X_{n+1}, \dots, X_k$ transcendental and algebraically independent over $K(X_1, \dots, X_n)$, there exist ω an extension of w to $K(X_1, \dots, X_k)$ that is symmetric with respect to $X_1, \dots, X_k$.

Once more we will use *reductio ad absurdum*. More specifically, let's suppose that $\text{symmdeg } w' = l > k$ and we notice two different sub-cases here, depending on w' being or not symmetric.

Case (II.a): w' is not symmetric so, for any $\{X_{j_1}, \dots, X_{j_l}\} \subset \{X_1, \dots, X_{n+r}\}$ we get that $w'|_{K(X_{j_1}, \dots, X_{j_l})}$ is symmetric with respect to $X_{j_1}, \dots, X_{j_l}$. In particular, $\omega = w'|_{K(X_1, \dots, X_l)}$ is symmetric with respect to $X_1, \dots, X_l$ and extends w to $K(X_1, \dots, X_l)$, so the maximality of k is contradicted.

Case (II.b): w' is symmetric hence, for any $X_{n+r+1}, \dots, X_l$ transcendental and algebraically independent over $K(X_1, \dots, X_{n+r})$, there exist ω' an extension of w' to $K(X_1, \dots, X_l)$ that is symmetric with respect to $X_1, \dots, X_l$, so ω' extends also w , preserving symmetry, but this contradicts again the maximality of k .

Case (III): Finally, in the last case, when $\text{symmdeg } w = \infty$, the inequality is obvious.

Q.E.D.

This result allows reducing the study of a symmetrically-closed valuation on $K(X_1, \dots, X_n)$ to the study of the most relevant restriction of it to $K(X_1, \dots, X_i)$, namely one of the restrictions u_i , where $i \leq n$.

Definition 4.6: Let w be an extension of v on K to $K(X_1, \dots, X_n)$. We will call *the most relevant restriction* (symmetrically-wise) of valuation w the following:

$$u_i := w|_{K(X_1, \dots, X_i)}$$

where i is the smallest index such that $1 \leq i \leq n$ and $\text{symmdeg } u_i = \text{symmdeg } w$.

Let's remark the fact that this definition is consistent also with the case when w is not symmetric, since $\text{symmdeg } w$ represents the number of indeterminates that, taken in any combination out of $X_1, \dots, X_n$, give equivalent restrictions. We may now observe a couple of things related to the notion of the most relevant restriction of a valuation.

Observation 4.7: If u_i is the most relevant restriction of w then the most relevant restriction of u_i is, obviously, u_i itself.

Observation 4.8: If u_i is the most relevant restriction of w then $i \leq \text{symmdeg } w$. Indeed, either w is symmetric and, according to Observation 4.3, $\text{symmdeg } w \geq n \geq i$ or w is asymmetric and, in this case, $\text{symmdeg } u_i = \text{symmdeg } w$ is the largest k , with $1 \leq k < n$, such that, for any set $\{X_{i_1}, \dots, X_{i_k}\} \subset \{X_1, \dots, X_n\}$, we are guaranteed that $w|_{K(X_{i_1}, \dots, X_{i_k})}$ is symmetric with respect to $X_{i_1}, \dots, X_{i_k}$, which includes the u_i case.

Observation 4.9: When $n \geq 2$ and w is symmetrically-open then the most relevant restriction of w is u_2 , since both have the symmetry degree equal to infinity.

5. CLASSES OF SYMMETRICALLY-CLOSED VALUATIONS

In the case of symmetrically-open valuations we discovered 7 classes of such valuations, these being the only possible ones, as a result of Theorem 3.6. We will attempt a similar classification for the symmetrically-closed ones. First, we will discuss a class of valuations that have a minimal most relevant restriction.

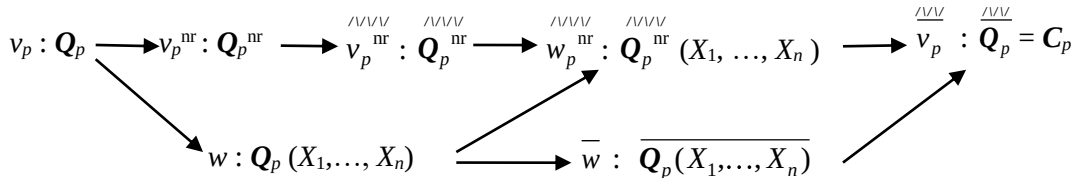
Proposition 5.1: Let $K = \mathbb{Q}_p$ the field of p -adic numbers, with p prime and $\mathbb{Q}_p^{\text{nr}}$ the maximal unramified extension of $\mathbb{Q}_p$. The valuation $v = v_p$ on $\mathbb{Q}_p$, extended to v_p^{nr} on $\mathbb{Q}_p^{\text{nr}}$, induces a norm on $\mathbb{Q}_p^{\text{nr}}$ which allows the completion of the latter to the field $L = \overline{\mathbb{Q}_p^{\text{nr}}}$.

Let $\overline{v_p^{\text{nr}}}$ be the extension of v_p^{nr} to $\overline{\mathbb{Q}_p^{\text{nr}}}$. Let E be an Eisenstein polynomial over L of degree n :

$$E = Y^n + a_{n-1}Y^{n-1} + \dots + a_0$$

where we chose $a_0, a_1, \dots, a_{n-1} \in L$ that are algebraically independent over $\mathbb{Q}_p$, having the valuations $v_p^{\text{nr}}(a_0) = 1$ and $v_p^{\text{nr}}(a_i) \geq 1$ for $i \geq 1$. Let $X_1, \dots, X_n$ be the roots of E over L .

Let's consider $\overline{v_p}$ the extension of v_p to $\overline{\mathbb{Q}_p}$, as depicted in the diagram below:



Finally, let $w = w_p^{\text{nr}} \Big|_{\mathbf{Q}_p(X_1, X_2)}$ be the restriction of w_p^{nr} to $\mathbf{Q}_p(X_1, \dots, X_n)$.

Then $\text{symmdeg } w = 2$, with respect to $X_1, \dots, X_n$, and the most relevant restriction of w is the restriction $u_2 = w|_{K(X_1, X_2)}$.

PROOF:

First, to be noted that $L = \mathbf{Q}_p^{\text{nr}}$ has an infinite transcendence degree over $\mathbf{Q}_p$ and the extensions $L \rightarrow L(X_i)$ are totally ramified extensions. Since the minimal monic polynomial of $X_1, \dots, X_n$ is E , whose coefficients are algebraically independent over $\mathbf{Q}_p$, it follows that $X_1, \dots, X_n$ are transcendental and algebraic independent over $\mathbf{Q}_p$, so they may be considered indeterminates for the extension w over v .

It is easy to see that w is symmetric with respect to $X_1, \dots, X_n$. Indeed, if it weren't, considering the permutation $\pi \in S_n$ and defining πw by

$$\pi w(f(X_1, \dots, X_n)) = w(f(X_{\pi(1)}, \dots, X_{\pi(n)}))$$

for all f in $\mathbf{Q}_p(X_1, \dots, X_n)$, we get that $\pi w \neq w$ but πw also extends v_p to $\mathbf{Q}_p^{\text{nr}}(X_1, \dots, X_n)$, contradicting the uniqueness of v_p in $\mathbf{Q}_p^{\text{nr}}(X_1, \dots, X_n)$.

The unique extension of w , say $\overline{w}$, to $\overline{\mathbf{Q}_p(X_1, \dots, X_n)}$ cannot extend the symmetry of w because, if it did, we would have, for any $b \in \overline{\mathbf{Q}_p}$:

$$\begin{aligned} \overline{w}(X_1 - b) &= \frac{1}{n} \overline{w}((X_1 - b) \cdot \dots \cdot (X_n - b)) = \\ &= \frac{1}{n} \overline{v}_p(b^n + a_{n-1}b^{n-1} + \dots + a_0) \end{aligned}$$

therefore, for a sequence $\{b_s\}_s$ in $\overline{\mathbf{Q}_p}$ with $\lim_s b_s = X_2 \in \overline{\mathbf{Q}_p}$, we would get:

$$\overline{w}(X_1 - X_2) = \lim_s \overline{w}(X_1 - b_s) = \infty.$$

According to Observation 3.3, we conclude that w is not symmetrically-open and cannot be extended further to a simple transcendent extension, $\mathbf{Q}_p(X_1, \dots, X_n, X_{n+1})$. Thus, we proved that $\text{symmdeg } w = n$ and the most relevant restriction of w is $u_2 = \overline{w}|_{K(X_1, X_2)}$ because $u_1 = \overline{w}|_{K(X_1)}$ is, trivially, symmetrically-open so it has an infinite symmetry degree.

Q.E.D.

We will now move to the other case, of the maximal most relevant restriction of a valuation. This is easy to verify for an extension that is Gaussian up to $K(X_1, \dots, X_{n-1})$, then having a r.a.f.-extension at the last intermediary extension. In this setup, we get the following result.

Proposition 5.2: Let w be an extension of v on K to $K(X_1, \dots, X_n)$ such that its restriction to $K(X_1, \dots, X_{n-1})$ is a Gaussian extension and the extension further to K to $K(X_1, \dots, X_n)$ is a r.a.f.-extension defined by the minimal pair $(-X_1 - \dots - X_{n-1}, \lambda)$, with $\lambda \notin G_{n-1}$.

Then $\text{symmdeg } w = n$ and the most relevant restriction of w is w itself.

PROOF:

From the hypothesis, for any $F \in K[X_1, \dots, X_n]$ written as:

$$F = \sum_{(i_1, \dots, i_n) \in I} a_{i_1, \dots, i_n} \cdot X_1^{i_1} \cdot \dots \cdot X_{n-1}^{i_{n-1}} \cdot (X_1 + \dots + X_n)^{i_n}, \text{ with } a_{i_1, \dots, i_n} \in K$$

where I is a finite set of n -uples of indices, we have:

$$w(F) = \inf_{(i_1, \dots, i_n) \in I} \left(v(a_{i_1, \dots, i_n}) + i_n \cdot \lambda \right).$$

Let's prove, first, that w is symmetric. According to [15, Lemma 3.1], in order to prove that w is symmetric it is sufficient to prove that w is symmetric with respect to X_i, X_n for each index i with $1 \leq i \leq n-1$, in fact it is sufficient to prove only that w is symmetric with respect to the pair X_{n-1}, X_n , the other pairs behaving similarly. Therefore, let $\pi \in S_n$ that inverts n with $n-1$, denote by $S = X_1 + \dots + X_n$ and rewrite F as:

$$\pi F = \sum_{(i_1, \dots, i_n) \in I} a_{i_1, \dots, i_n} \cdot X_1^{i_1} \cdot \dots \cdot X_{n-2}^{i_{n-2}} \cdot X_n^{i_{n-1}} \cdot (X_1 + \dots + X_{n-2} + X_n + X_{n-1})^{i_n} =$$

$$\begin{aligned}
 &= \sum_{(i_1, \dots, i_n) \in I} a_{i_1, \dots, i_n} \cdot X_1^{i_1} \cdot \dots \cdot X_{n-2}^{i_{n-2}} \cdot (S - X_1 - \dots - X_{n-1})^{i_{n-1}} \cdot (X_1 + \dots + X_n)^{i_n} = \\
 &= \sum_{(i_1, \dots, i_n) \in I} \sum_{j=0}^{i_{n-1}} a_{i_1, \dots, i_n} \cdot C_{i_{n-1}}^j X_1^{i_1} \cdot \dots \cdot X_{n-2}^{i_{n-2}} \cdot (-X_1 - \dots - X_{n-1})^{i_{n-1}-j} \cdot S^{i_n+j}
 \end{aligned}$$

By using an index k as equaling $i_n + j$ in all terms we may continue the expansion (formally considering null extra $a_{i_1, \dots, i_n}$ coefficients that may appear in the expansion):

$$\begin{aligned}
 \pi F &= \sum_{(i_1, \dots, i_n) \in I} \sum_{j=0}^{i_{n-1}} a_{i_1, \dots, i_n} \cdot C_{i_{n-1}}^j X_1^{i_1} \cdot \dots \cdot X_{n-2}^{i_{n-2}} \cdot (-X_1 - \dots - X_{n-1})^{i_{n-1}-j} \cdot S^{i_n+j} = \\
 &= \sum_{k \geq 0} \sum_{(i_1, \dots, i_{n-2}, \bullet, \bullet) \in I} \sum_{i_1 \geq k} \sum_{i_n=0}^k a_{i_1, \dots, i_n} \cdot C_{i_{n-1}}^{k-i_n} X_1^{i_1} \cdot \dots \cdot X_{n-2}^{i_{n-2}} \cdot (-X_1 - \dots - X_{n-1})^{i_{n-1}+i_n-k} \cdot S^k
 \end{aligned}$$

For a fixed k , let's put:

$$F_k = \sum_{(i_1, \dots, i_{n-2}, \bullet, \bullet) \in I} \sum_{i_{n-1} \geq k} \sum_{i_n=0}^k (-1)^{i_{n-1}+i_n-k} \cdot a_{i_1, \dots, i_n} \cdot C_{i_{n-1}}^{k-i_n} X_1^{i_1} \cdot \dots \cdot X_{n-2}^{i_{n-2}} \cdot (X_1 + \dots + X_{n-1})^{i_{n-1}+i_n-k}$$

which is a polynomial in $K(X_1, \dots, X_{n-1})$. From [8] we know that $(-X_1 - \dots - X_{n-2}, 0)$ is a minimal pair of definition for u_{n-1} , just like $(0, 0)$, since $-X_1 - \dots - X_{n-2}$ has degree 0 over $K(X_1, \dots, X_{n-2})$ and verifies:

$$u_{n-1}(X_{n-1} - (-X_1 - \dots - X_{n-2})) = u_{n-1}(X_{n-1} - 0) = 0$$

By using the infimum formula with this new minimal pair for u_{n-1} we get:

$$w(F_k) = \inf_{\substack{(i_1, \dots, i_{n-2}, \bullet, \bullet) \in I \\ i_{n-1} \geq k \\ 0 \leq i_n \leq k}} \left(v(a_{i_1, \dots, i_n}) + v(C_{i_{n-1}}^{k-i_n}) \right)$$

We now remember that k was used instead of $i_n + j$ and write:

$$\begin{aligned}
 w(\pi F) &= \inf_{\substack{k \geq 0 \\ (i_1, \dots, i_{n-2}, \bullet, \bullet) \in I \\ i_{n-1} \geq k \\ 0 \leq i_n \leq k}} \left(v(a_{i_1, \dots, i_n}) + v(C_{i_{n-1}}^{k-i_n}) + k \cdot \lambda \right) = \\
 &= \inf_{\substack{(i_1, \dots, i_n) \in I \\ 0 \leq j \leq i_{n-1}}} \left(v(a_{i_1, \dots, i_n}) + v(C_{i_{n-1}}^j) + (i_n + j) \cdot \lambda \right)
 \end{aligned}$$

By noticing that:

$$w(X_n - X_1) = w'(X_1 + \dots + X_n - 2X_1 - \dots - X_{n-1}) \geq \inf(\lambda, 0)$$

with $\lambda \notin G_{n-1}$, we get that $\lambda > 0$. This means that, for a fixed $(i_1, \dots, i_n) \in I$, the smallest quantity using this n -uple must have $j = 0$. Hence, we get:

$$w(\pi F) = \inf_{(i_1, \dots, i_n) \in I} \left(v(a_{i_1, \dots, i_n}) + i_n \cdot \lambda \right) = w(F)$$

which proves that w is symmetric.

Now, let's assume that w might be extended to w' on $K(X_1, \dots, X_{n+1})$, with X_{n+1} transcendental over $K(X_1, \dots, X_n)$, also symmetric. Then we would have:

$$\begin{aligned} w'(X_{n+1} - X_n) &= w'(X_{n+1} + X_1 + \dots + X_{n-1} - X_1 - \dots - X_{n-1} - X_n) \geq \\ &\geq \inf(w'(X_1 + \dots + X_{n-1} + X_{n+1}), w(X_1 + \dots + X_{n-1} + X_n)) = \\ &= \inf(\lambda, \lambda) > 0 = w(X_1 - X_2) \end{aligned}$$

which would contradict the symmetry of w' . This means that $\text{symmdeg} = n$ and the most relevant restriction of w is w itself, as u_{n-1} is symmetrically-open, so $\text{symmdeg } u_{n-1} = \infty$.

6. Conclusion

We studied the symmetrically-closed extensions through the lens of the freedom degree and the newly defined notions of symmetry degree and the most relevant restriction (symmetrically-wise, i.e. the restriction to the minimal number of indeterminates that preserves the symmetry degree).

We know have a larger classification of symmetric extensions w , of a valuation v on K to $K(X_1, \dots, X_n)$:

- symmetrically-open extensions, that have $\text{symmdeg } w = \infty$ and their most relevant restriction being u_2 ; these are classified further in two sub-types:
 - (I) generalization of a Gauss extension;
 - (II) limit of a convergent sequence of extensions of type (I);
- extensions with $\text{symmdeg } w \geq n$, $\text{symmdeg } w < \infty$, their most relevant restriction being u_2 and $\text{freedeg } w = n$;

- extensions with $\text{symmdeg } w = n < \infty$ and $\text{symmdeg } u_{n-1} = \infty$, hence their most relevant restriction is w itself and $\text{freedeg } w \leq 3$; for these, the last intermediary extension closes the chain of symmetrically-open extensions.

Finding the other possible classes of symmetrically-closed extensions would complete the classification of the symmetric extensions in general and would bring us one step closer to closing the chapter of the extensions of valuations to $K(X_1, \dots, X_n)$.

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