ASPECTS CONCERNING NONLINEARITY IN ECONOMETRIC MODELING

Cristina COCULESCU *[1]
Adam ALTĂR-SAMUEL [2]

ABSTRACT

In this paper, we present certain points of view regarding nonlinearity in the construction of econometric models, as well as its implications on parameter estimation. The research analyzes different scenarios encountered in economic practice and the appropriate methods of approach found in the econometric literature. In this context, we also present the authors' personal viewpoints regarding modeling of nonlinear economic relationships, substantiated by practice in modeling and simulation of economic phenomena and processes. Classifying nonlinear relationships is of importance both for short-term estimation purposes and especially for the analysis of the precision of medium to long term predictions. There is a class of nonlinear models that can be approximated well by linear models, easily approachable using the OLS (Ordinary Least Squares) method. When we opt for such an approach, analyzing how the residual variable (the error) fits within the model is essential (meaning that linearization can be hindered by the additive or multiplicative form in which the error is expressed). Also, practice has shown that medium to long term evolutions can sometimes be nonlinear even though, for short-term predictions, a linear model has proven acceptable.

Keywords: Econometric Modeling, Linear/Nonlinear, Estimation

Acknowledgement: Financial Support from PN-II-ID-PCE-2011-3-1054 research project entitled "Uncertainty, Complexity and Financial System Stability" is gratefully acknowledged.

1. Nonlinear Relations in Economics

In economic studies, using regression for measuring the influence on the evolution of an economic process of one or several factors is of great interest both for analysis and for forecasting. The concept of regression was proposed by English biologist Fr. Galton who, studying the relation between boys' height at army incorporation and their parents' height (father's) showed the tendency of reverting boys' height to average height (of male persons) whenever the father's height was much under or over average. Therefore, we have a "regression" in the sense of "reverting to the mean", on the one hand; on the other hand, "regression" means searching and quantifying a relation between a variable (boy's height) and another variable (his father's height). Such a relation is expressed by an equation in which *y* (dependent variable) depends on one or several explanatory variables;

[1] *Corresonding author. Associate Professor, PhD. Department of Computer Science, Statistics and Mathematics, Romanian – American University. 1B, Expozitiei Blvd., district 1, code 012101, Bucharest, Romania, E-mail: cristina_coculescu@yahoo.com
[2] Teaching Assistant, Department of Computer Science, Statistics and Mathematics, Romanian-American University, 1B, Expozitiei Blvd., district 1, code 012101, Bucharest, Romania, E-mail: adamalt@gmail.com

the regression equation can be written as $y = f(x)$. This relation is not deterministic, in the sense that it is usually inferred by studying a great number of situations (statistical data); hence, we deal with a statistical regression.

The basic element of statistical regression is the regression function. The shape of the regression function can be suggested by economic theory, but, in most cases, it is empirically established and, for this reason, graphical sketch of the points cloud (scatter diagram) is the main starting point in the choice of a linear or a nonlinear function.

The term "linear" can be understood as having the following meanings (Iacob and Tănăsoiu, 2005), (Pecican, 2003):

a) linear with respect to variables and parameters: in this case, the variables and parameters are at the power 1 (one) ($y = a_0 + a_1x_1 + \cdots + a_kx_k$);
b) nonlinear with respect to variables, but linear with respect to parameters, case when at least one of the explanatory variables is at an exponent different from 1, e.g. ($y = a_0 + a_1x + a_2x^2$; $y = ax^{\alpha}z^{\beta}$, for $\alpha \neq 1, \beta \neq 1$;
c) nonlinear with respect to parameters, but linear with respect to variables, case when at least one parameter is at an exponent different from 1, e.g. ($y = a + a^2x$; $y = \sqrt{a}x$);
d) nonlinear with respect to variables, and parameters, case when at least one variable and at least one parameter are at a power different from 1, e.g. ($y = a^2x + ax^2$).

Note that if products or ratios of variables or parameters are present in the regression function, then the model is nonlinear.
Such a classification is of interest both for the description of the analyzed process and for parameter estimation, since the estimation method mostly used in econometrics, the OLS (Ordinary Least Squares) method, is well-adapted only to the linear case (Ebâncă, 1994).
Nonlinear variants impossible to study by the method of least squares include nonlinear models either with respect to parameters or with respect to parameters and variables, or functional forms in which the residual variable is included in ways that prevent linearization. Many relations between economic variables in economic theory exhibit nonlinear shape. Nonlinearity in the evolution of a relatively great number of phenomena occurs especially in situations regarding (Pecican, 2003):

- nearness of certain limits over which "it is difficult to pass";
- nearness of a saturation level, with respect to which factor activity leaves the effect "indifferent";
- reaching and overtaking a certain critical point with respect to which a factor gives rise to an effect which is contrary to that generated initially and preserved until reaching the critical point.

We mention some variable pairs for which economic theory suggests such nonlinear relations:

- price – bought quantity (demand curve);
- price –supplied quantity (supply curve);

- wage per time unity – quantity of labour (curve of labour demand, curve of labour supply);
- interest rate –money demand or money supply (curve of money demand, curve of money supply);
- relative wage rate– unemployment rate (Phillips curve).

Other types of nonlinear evolution refer to:

- cost (variable or total) –output;
- GDP and its cyclic variation;
- Demand for a commodity during time and the succession of stages in the life of the commodity on the market (launching, growth, maturity, stagnation or decline).

Evolutions of logarithmic or semi-logarithmic types are characteristic for the variable pairs:

- production– production factor;
- current demand – permanent income;
- degree of satisfaction – quantity available.

Nonlinear evolutions admitting an extremum point:

- budget incomes (from income tax) – tax rate (Laffer curve);
- cost –output;
- evolution of marginal cost.

A special class of nonlinear models occurs in connection with financial data.
It is likely that many relationships in finance are intrinsically non-linear. As (Campbell, Lo and MacKinlay, 1997) state, the payoffs to options are non-linear in some of the input variables, and investors' willingness to trade off returns and risks are also non-linear. These observations provide clear motivations for consideration of non-linear models in a variety of circumstances in order to capture better the relevant features of the data. Linear structural (and time series) models are also unable to explain a number of important features common to much financial data, including:
● *Leptokurtosis* -- that is, the tendency for financial asset returns to have distributions that exhibit fat tails and excess peakedness at the mean.
● *Volatility clustering or volatility pooling* -- the tendency for volatility in financial markets to appear in bunches. Thus large returns (of either sign) are expected to follow large returns, and small returns (of either sign) to follow small returns. A plausible explanation for this phenomenon, which seems to be an almost universal feature of asset return series in finance, is that the information arrivals which drive price changes themselves occur in bunches rather than being evenly spaced over time.
● *Leverage effects* -- the tendency for volatility to rise more following a large price fall than following a price rise of the same magnitude.
Campbell, Lo and MacKinlay broadly define a non-linear data generating process as one where the current value of the series is related non-linearly to current and previous values of the error term $y_t = f(u_t, u_{t-1}, u_{t-2}, \dots)$ where u_t is an iid error term and f is a non-

linear function. According to Campbell, Lo and MacKinlay, a more workable and slightly more specific definition of a non-linear model is given by the equation $y_t = g(u_t, u_{t-1}, u_{t-2}, \dots) + u_t\sigma^2(u_{t-1}, u_{t-2}, \dots)$ where g is a function of past error terms only, and σ^2 can be interpreted as a variance term, since it is multiplied by the current value of the error.
Campbell, Lo and MacKinlay usefully characterize models with non-linear $g(\bullet)$ as being non-linear in mean, while those with non-linear $\sigma^2(\bullet)$ are characterized as being non-linear in variance. Models can be linear in mean and variance (e.g. the CLRM, ARMA models) or linear in mean, but non-linear in variance (e.g. GARCH models).

2. Estimation of Parameters in Case of Nonlinear Links between Variables

Parameter estimation is the basic problem in Econometrics. The quality of estimators, as well as the choice between estimation methods, rely on the following criteria:

- The degree of determination should be as large as possible;
- the distances (deviations) between empirical values y_i and values resulted from application of the model ($\hat{y}_i$) should be as small as possible, so that $\sum_i (y_i - \hat{y}_i)^2$ be minimized;
- the estimations obtained should be unbiased, consistent, efficient (all these characteristics have precise definitions in Mathematical Statistics);
- the cost of applying the estimation method is minimal.

The method that satisfies, under certain basic hypotheses (the so-called Classical Normal Linear Regression Model) all these criteria, is the OLS (Ordinary Least Squares) method. The founder of this method is considered C.F. Gauss, but the method was also proposed, independently, by the French mathematician A.M. Legendre. Important contributions to this method are due to the mathematicians P.S. Laplace, P.L. Chebyshev, A.A. Markov.

Practically, in cases of linearity with respect to parameters, packages of computer programs are available, which include numerical methods for founding the solutions (estimators which minimize the sum of squares of deviations). Although the methods included in such program packages are not identical, the estimation problem can be approached at the level of general principles and stages common to most procedures (Ebâncă, 1994). As concerns the properties of estimators of the nonlinear model, we note that estimations are considered asymptotically normally distributed and consistent.
Referring to the nonlinearity problem, Kmenta (Kmenta, 1971) asserts that the difference between the linear (usual) method of least squares and the same method in the nonlinear case, is due to the fact that, in the linear case, the estimator can be expressed as a linear function of disturbances (errors), what is not possible, in general, in the nonlinear case.
The fragility of estimation in the nonlinear case is caused by the procedure of finding the estimator, when the procedure of successive trials ("trial and error") is used. Such a procedure supposes assigning several values for a given parameter in the likelihood function, computing and comparing the corresponding values of the function, until we reach a value that does not improve after ulterior trials (attributions), so that it appears to be maximum (Andrei and Bourbonnais, 2009). The solution could be accepted if we knew

that the function is unimodular (has only one maximum) but, if it has several local maxima, then the trials can continue, in the hope of reaching a global maximum. When using a computer, the procedure is not difficult to implement, especially because the set of admissible values is, frequently, finite. We still have to solve the problem of the sensitivity of the estimation thus obtained with respect to errors in data set of the sample.

3. Implications of Nonlinearity on Estimation of Parameters of Econometric Models

Nonlinearity generates, concerning estimation, unwanted implications, caused by:

- the position of the perturbation in the structure of the model (it can appear either as term or as factor or as exponent) and, in connection with this, the validity of the supposition that the residual variable follows a normal law, of zero mean, finite and constant dispersion and the evolution of deviations (residuals) is not self-correlated;
- impossibility of obtaining analytical formulas for the estimators that minimize the sum of squared of deviations .

In what follows, we present some points of view about nonlinearity in building econometric models and its implications for parameter estimation.

So, Kmenta (Kmenta, 1971), referring to the relation between income (y) and age (x), pairs of variables frequently encountered in opinion polls among buyers, given by the equality:

$$y = a_0 + a_1 x + a_2 x^2 + u$$

showed that “linearization” through the replacement of variable x^2 with *z* leads to the equality:

$$y = a_0 + a_1 x + a_2 z + u$$

where the factors x and z have a certain degree of co-linearity, which diminishes the efficiency of estimations.

As concerns relations of the form $= e^{a+bx+u}$, we note that, when the factor (x) has a linear evolution in time (e.g .when x represents age, time), the mean effect $M(y)$ also evolves with a constant rate. For $x_{t+1} - x_t = 1$,

$$\frac{M(y_{t+1})}{M(y_t)} = \frac{e^{a+bx_{t+1}} \cdot M(e^{u_{t+1}})}{e^{a+bx_t} \cdot M(e^{u_t})} = e^b$$

A simple transformation, $\ln y = w$, leads to the linear form $w = a + bx + u$.
However, there are representations more difficult to linearize. For example, a relation of the type:

$$C_t = a \cdot P_t^b \cdot V_t^c + u_t$$

where C = consumption, V = income, P = price, which differs from the exponential function above by including the error (u) in additive form. Partial elasticities of such functions lead to parameters b and c , although a linear transformation cannot be used for estimation. When the residual variable follows a normal distribution, and the factors (P, V) do not depend on the error (u), we can use the maximum likelihood method; in this case

$$L = -(n/2)\ln 2\pi - (n/2)\ln \sigma^2 - \frac{1}{2\sigma^2} \cdot \sum \left(C - a \cdot P^b \cdot V^c\right)^2$$

Annulling the partial derivatives with respect to the parameters a, b, c and σ^2 leads to a four equation system, nonlinear with respect to the unknowns. Applying a procedure of the trial and error type, by trying several combinations of values for the parameters until the maximum likelihood is reached, could represent a solution (Kmenta, 1971) (Pecican, 2003).

Griffith refers to a relatively similar case, describing a consumption function of nonlinear form: $C_t = a_1 + a_2 \cdot V_t^b + u_t$ that can be modified, depending on what data suggest (e.g. assigning to parameter b acceptable values such as $b = \frac{1}{2}$, $b = 2$ or $b = -1$); this makes it possible to create variants of the model, transformable into linear representations, for which the method of least squares can be useful for estimation (Griffiths, Carter and Judge, 1993).
An attempt to generalize models in which the effect variable (Y) depends on one or more factors (X) would place the linear model as a particular case of a regression model in general form, expressed by the equality:

$$Y = h(X, A) + U$$

where A – parameters vector; X – matrix of factorial variables.

Referring to the cases represented by expressions as $y_i = a \cdot X_i^b \cdot Z_i^c \cdot e_i^u$ (nonlinear case which can be made linear) and $y = a + b \cdot e^{cX} + e^{cX} + u$ (nonlinear case), Greene (Greene, 2000) considers that the element which distinguishes the linear regression model from the nonlinear one, is the method of parameter estimation.
In the second case, the solutions for parameters (obtained from the equations resulted by annulling the partial derivatives of order 1) are not explicit, so that using an iterative procedure for obtaining the estimations could represent a way of solving.
Greene (Greene, 1992) concludes that the nonlinear regression model could be characterized (defined) as being a model for which the so-called normal equations (obtained by annulling the partial derivatives of the sum of squared residuals with respect

to every unknown, when using the least squares method) are nonlinear functions of parameters .
The same author remarks that in such cases, impossible to tackle by using the least squares method directly, it is useful to consider the linear approximation of the Taylor series associated to $h(X, A)$ around particular values A^* (derived from estimations through the least squares method or "trial and error") for the elements of the vector A of parameters. Such a function, being linear, is tractable using the least squares method for obtaining estimations which will replace the initial settings A^*. The process is resumed so that with every iteration we can consider that we update the prior set of estimated parameters, until the differences resulted in successive steps become small enough to consider the convergence achieved.

Referring to this procedure, Kmenta (Kmenta, 1971) considers that it can be extended for functions of several variables. Thus, in the two-factor case, the procedure allows getting linear approximations for the product or rate computed for two variables.

A possibility to characterize the general representation of the regression model through the appropriate type of function (linear/nonlinear) is through the so-called Box-Cox transformation. In this transformation, the independent variable x is expressed as a function of the parameter λ, as $x^{(\lambda)} = \left(x^{\lambda} - 1\right) / \lambda$. In principle, every regression factor can be thus transformed, by setting values for the parameter λ.
For such a value, the model becomes:

$$y = a + \sum_{j=1}^{k} a_j x_j^{(\lambda)} + u$$

When $\lambda = 1$, the model is linear; for $\lambda = -1$ the equation includes the inverse of the variable *x*; for $\lambda = 0$ a semi- logarithmic model (or log-lin) arises. For several other values for λ, different nonlinear forms are obtained.
A representation having a greater degree of generalization is proposed by Johnston (Johnston, 1991); it concerns both variables *y* and *x*:

- for $\lambda \neq 0$, $y^{(\lambda)} = \left(y^{\lambda} - 1\right) / \lambda$, $x^{(\lambda)} = \left(x^{\lambda} - 1\right) / \lambda$;
- for $\lambda = 0$, $y^{(\lambda)} = \ln y$, $x^{(\lambda)} = \ln x$.

4. Types of Nonlinear Models

Consider, for example, the following models:

$$y = ax^b e^u \quad (1)$$
$$y = ax^b u \quad (2)$$
$$y = ax^b + u \quad (3)$$
$$y = bx + b^2 z + u \quad (4)$$

Model (1) can be linearized by the transformation: $\ln y = \ln a + b \ln x + u$, which allows using the OLS (Ordinary Least Squares) method without troubles. The Model (2) can also be linearized: $\ln y = \ln a + b \ln x + \ln u$, but, in order to apply the OLS (Ordinary Least Squares) method, we must suppose that $\ln u$ has normal distribution of zero mean and finite dispersion ($\ln u \in N(0, \sigma^2)$).Therefore, under this hypothesis, the residual variable "u" would have to follow a log-normal distribution. In practice, this is not always true.
The model (3) in the form $y = ax^b + u$ cannot be linearized by using logarithm or any kind of algebraic transformations. To linearize this model, we can use an iterative process that will be described in what follows. Neither can model (4) be approached using the OLS (Ordinary Least Squares) method.
In cases of nonlinearity with respect to parameters (e.g. cases 3 and 4), an iterative procedure is suggested for linearization; this supposes the following stages:

Stage 1. We assign an initial value for the parameter, either depending on the nature of the economic problem or suggested by the level of some statistical indicators (e.g. the correlation coefficient);

Stage 2. We calculate the sum ($\sum_t u_t^2$), for the parameter value set at stage 1;

Stage 3. We continue changing the estimation obtained until this stage, aiming at minimizing the sum of squared residuals ($\sum_t u_t^2$);

Stage 4. The sum of squared residuals obtained in stage 3 is compared to the sum obtained in stage 2. If the difference between the two sums is significant, the procedure is resumed, considering as basis for comparison the last sum obtained.

Thus, we seek convergence of the sum of squared residuals to a stable level. When this aim is reached, the last value estimated for the parameter represents the solution.
In this context, the following problem occurs: Is the *solution we finally chose a local or a global minimum*? To answer this question, we proceed by successive trials, starting from different initial values of the parameter. If the solutions converge towards the same level (obtained by using the algorithm above), then we can consider that the solution is a global minimum. Such procedures of searching optimum solutions by using successive iterations are included into program packages frequently used in econometric applications (Ebâncă, 1994) (Pecican, 2003).
The non-linear models presented above, non-approachable through the OLS (Ordinary Least Squares) method, incorporate non-linearity either with respect to parameters or with respect to parameters and to variables, or structures in which the way of incorporating the residual variable makes linearization difficult. Therefore, the way in which the residual variable is included in the model is important in view of the possibility of linearizing the function.

We also mention the possibility of using non-linear models, impossible to estimate by using the OLS (Ordinary Least Squares) method, such as:

- the production function: $y = ax^b z^c + u$

- consumption functions: $y = a + bx^c + u$
  $y = ax + a^2 z + u$
  $y = a + \sqrt{b}x + u$;

logistic function of the form $y = \frac{a}{1+e^{b-ct}}$, used especially for the study of economic and demographic phenomena, where:

$t = 1, 2, 3, \dots, n$;
a = parameter that indicates an upper limit for growth (saturation level);
b = parameter (e^b shows the position of the curve with respect to the time axis);
c = parameter representing the slope of the curve; $a, b, c > 0$.

The estimation of the parameters a, b and c of the logistic function can be achieved by using the method of equidistant points. This method supposes dividing the time series in three equal parts and, for each part, defining a characteristic value (as a rule, the median value): $y(1), y(2), y(3)$ (Ebâncă, 1994).

From relation: $y = \frac{a}{1+e^{b-ct}} \Rightarrow e^{b-ct} = \frac{a-y}{y} \Rightarrow b - ct = \ln\frac{a-y}{y}$, by replacing the variable *y* with the characteristic values $y(1), y(2), y(3)$, we obtain:

- for time $t = 0 \Rightarrow b = \ln\frac{a-y(1)}{y(1)}$ (5)

- for time $t = k \Rightarrow b - ck = \ln\frac{a-y(2)}{y(2)}$ (6)

- for time $t = 2k \Rightarrow b - 2ck = \ln\frac{a-y(3)}{y(3)}$ (7)

where: *k* = distance between characteristic values.
The relations (5), (6), (7) form a system of three equations with three unknowns, whose solution provides the estimators of the parameters *a*, *b*, *c*:

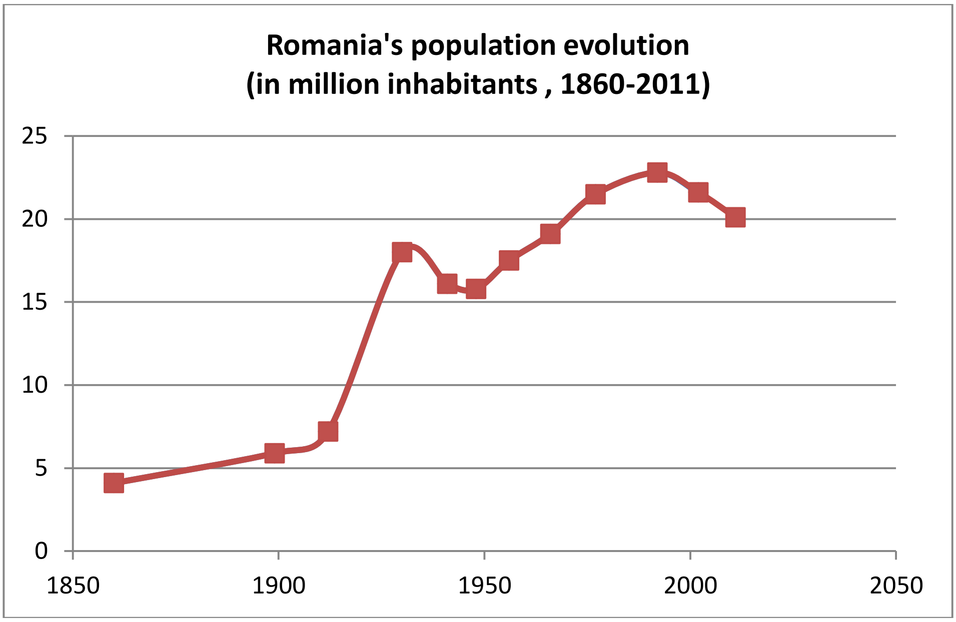
$$\hat{a} = \frac{2y(1)y(2)y(3)-y(2)^2[y(1)+y(3)]}{y(1)y(3)-y(2)^2}, \ \hat{b} = \ln\frac{\hat{a}-y(1)}{y(1)}, \ \hat{c} = \frac{\hat{b}-\ln\frac{\hat{a}-y(3)}{y(3)}}{2k}.$$

5. Example

Data about Romania's number of inhabitants, obtained through population censuses between the years 1860 and 2011, are presented in the table below.

Year	1860	1899	1912	1930	1941	1948	1956	1966	1977	1992	2002	2011
Population (millions)	4.1	5.9	7.2	18	16.1	15.8	17.5	19.1	21.5	22.8	21.6	20.1

Source: National Statistical Institute



Data fitting on the basis of the logistic function gave the estimations: $\hat{a} = 23{,}73$, $\hat{b} = 0{,}96$, $\hat{c} = 0{,}45$.

$$\Longrightarrow \quad \hat{y} = \frac{\hat{a}}{1 + e^{\hat{b} - \hat{c}t}} = \frac{23.73}{1 + e^{0{,}96 - 0{,}45t}}$$

Therefore, if the evolution rule observed in the past will be maintained, Romania's population will stabilize, corresponding to parameter *a*, around the level of 23.73 million inhabitants.

6. Conclusions

Through its methods and formal representations, Econometrics tries to reflect in a realistic, although simplified way, different influences which run through the economy considered as a whole.
Economic theory signals the directions of influences, often also the appropriate function which describes the economic process, Econometrics having the task of empirical evaluation of the realism and validity of such influences, but especially their numerical characterization, in a given economic context, at a given time.

When we choose to use the least squares method, it is very important to consider the way how the residual variable (error) is included into model (in the sense that, in some variants, linearization is prevented exactly by the multiplicative or additive way in which the error is included).
Practically, in cases of linearity with respect to parameters, packages of computer programs are available, which include numerical methods for founding the solutions (estimators which minimize the sum of squares of deviations). Although the methods included in such program packages are not identical, the estimation problem can be approached at the level of general principles and stages common to most procedures (Ebâncă, 1994). As concerns the properties of estimators of the nonlinear model, we note that estimations are considered asymptotically normally distributed and consistent.
The diversity of nonlinear links is important both from the viewpoint of estimation and in what concerns precision of medium–term and long-term forecasts. Thus, some nonlinear models can be transformed into linear models, tractable by the least squares method for parameter estimation; on the other hand, medium or long term evolution often risks to become nonlinear even in cases when, for the available data, a linear form of the model was accepted.

REFERENCES

[1] Andrei, T. and Bourbonnais, R. (2009) *Econometrie*, Bucuresti: Editura Economica.
Brooks, C. (2008) *Introductory Econometrics for Finance*, 2nd edition, Cambridge: Cambridge University Press.
[2] Campbell, J.Y., Lo, A.W. and MacKinlay, A.C. (1997) *The Econometrics of Financial Markets*, Princeton, N.J.: Princeton University Press.
[3] Ebâncă, D. (1994) *Metode de calcul numeric*, Craiova: Editura Sitech.
[4] Greene, W. (1992) *Econometric Analysis*, New York: Macmillan.
[5] Greene, W. (2000) *Econometric Analysis*, Prentice Hall.
[6] Griffiths, W., Carter, H. and Judge, G. (1993) *Learning and Practicing Econometrics*, Wiley Publishers.
[7] Gujarati, D.N. (1995) *Basic Econometrics*, New York: McGraw Hill.
[8] Iacob, A.I. and Tănăsoiu, O. (2005) *Modele econometrice*, Bucharest: Editura A.S.E.
[9] Johnston, J. (1991) *Econometric Methods*, New York: McGraw Hill.
[10] Kmenta, J. (1971) *Elements of Econometrics*, New York: Macmillan.
[11] Pecican, E.S. (2003) *Econometria pentru economisti*, Bucharest: Editura Economica.
[12] Voineagu, V. and Tițan, E. (2003) *Teorie și practică pentru economiști*, Bucharest: Editura Economica.