INITIAL VALUE PROBLEMS FOR NONLINEAR DIFFERENTIAL EQUATIONS SOLVED BY DIFFERENTIAL TRANSFORM METHOD

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ABSTRACT

The notion of differential transform was firstly introduced and applied to electrical circuits by J. Zhou, [4]. In the present paper we apply the differential transform method to solve initial values problems for nonlinear differential equations. This method can be easy implemented on the computer because of the recurrence relation established. The chosen example in this paper is proved to have an exact solution. A Maple program is also included. The nonlinear differential equation solved here for a initial value problem was also solved in the papers [2] and [3] for some boundary value problems.

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DIFFERENTIAL TRANSFORM METHOD

In this section we shall give some basic theorems of the one-dimensional differential transform method.

For x_0 a fixed real number, the differential transform F_{x_0} of an infinitely differentiable function *u (x)* is defined as being the following numerical sequence:

$$F_{x_0}(u(x)) = \left(\frac{1}{k!}u^{(k)}(x_0)\right) = (U(k)).$$

The inverse differential transform $F_{x_0}^{-1}$ associate to a numerical sequence (a_k) the function $u(x)$ defined as follows:

$$u(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \text{, with } a_k = \frac{1}{k!}u^{(k)}(x_0).$$

So, $$F_{x_0}^{-1}((a_k)) = \sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

The transformation is linear and has the next properties:

$$F_{x_0}\left(u^{(m)}(x)\right) = \left(\frac{(m+k)!}{k!}U(m+k)\right)$$

$$F_{x_0}(u(x)v(x)) = (U(k)) * (V(k)) = \left(\sum_{k=0}^{n} U(k)V(n-k)\right)$$

$$F_{x_0}\left((x - x_0)^n\right) = (\delta(k-n))$$

where $$\delta(k-n) = \begin{cases} 1 & , \quad k = n \\ 0 & , \quad k \neq n \end{cases}$$

INITIAL VALUE PROBLEM FOR NONLINEAR DIFFERENTIAL EQUATIONS

In this paper we investigate the following initial value problem for nonlinear differential equations :

u”(x) + F(x,u(x),u”(x)) = 0, $0 \le x \le 1$,
u(0) =a, u’(0) =b.

Applying the differential transform on the equation, and using its above mentioned properties, it results a recurrence relation of the form:

U(k+2)=G(U(0),U(1),…,U(k+1)),

with the initial values U(0)=u(0) =a, U(1) =u’ (0) =b, and G : [0, 1] × [0,∞) → [0,∞) a continuous function.

EXAMPLE

We consider the initial value problem composed from nonlinear differential equation

u”(x) + 2(u’(x))^2 + 8u(x) = 0, $0 \le x < \infty$,
and the following initial values: u(0) = 0, u’(0) = 1.

Using the method presented above, it results the following recurrence relation:

$$U(k+2) = -\frac{1}{(k+1)(k+2)}\left[2\sum_{l=0}^{k}(l+1)(k-l+1)U(l+1)U(k-l+1) + 8U(k)\right].$$

For k=0 we have : $U(2) = -\frac{1}{2}\left[2(U(1))^2 + 8U(0)\right] = -1$,

for k=1 : $U(3) = -\frac{1}{6}\left[2(2U(1)U(2) + 2U(2)U(1)) + 8U(1)\right] = -\frac{1}{6}(-8+8) = 0$,

and for k=2 : $U(4) = -\frac{1}{6}\left[2(3U(1)U(3) + 4U(2)^2 + 3U(3)U(1)) + 8U(2)\right] = -\frac{1}{12}(8-8) = 0$.

For $k \ge 3$, we suppose that $U(l) = 0$, $l = 3,\ldots,k+1$, and then

$$U(k+2) = -\frac{1}{(k+1)(k+2)}\left[2(1\cdot(k+1)U(1)U(k+1) + 2\cdot kU(2)U(k) + 3\cdot(k-1)U(3)U(k-1) + \right.$$
$$\left. + \cdots + (k-1)\cdot 3\cdot U(k-1)U(3) + k\cdot 2\cdot U(k)U(2) + (k+1)\cdot 1\cdot U(k+1)U(1)) + 8U(k)\right] = 0.$$

From the induction axiom, we have that $U(k) = 0$, $k = 3,4,\ldots$

The exact solution of the initial value problem is:

$$u(x) = \sum_{k=0}^{\infty} U(k)x^k = U(0) + U(1)x + U(2)x^2 = x - x^2,\ \forall x \in [0,\infty).$$

MAPLE PROGRAM

The current program doesn't solve the nonlinear differential equation.
We use the recurrence relation established in the example to see that we have the correct answer.

> **restart;**

The equation is :

> **Diff(u(x),x$2)+2*(Diff(u(x),x))^2+8*u(x)=0;**

$$\left(\frac{d^2}{dx^2}\mathrm{u}(x)\right)+2\left(\frac{d}{dx}\mathrm{u}(x)\right)^2+8\,\mathrm{u}(x)=0$$

The initial values are :
> **u(0)=0;**
> **Diff(u(0),x)=1;**

$$\mathrm{u}(0)=0$$

$$\frac{d}{dx}\mathrm{u}(0)=1$$

The transformed initial values :
> **U(0):=0;**
> **U(1):=1;**
The recurrence relation is :

> **U(k+2):=-1/((k+1)*(k+2))*(2*Sum((l+1)*(k-l+1)*U(l+1)*U(k-l+1),l=0..k)+8*U(k));**

$$\mathrm{U}(k+2):=-\frac{2\left(\sum_{l=0}^{k}(l+1)\,(k-l+1)\,\mathrm{U}(l+1)\,\mathrm{U}(k-l+1)\right)+8\,\mathrm{U}(k)}{(k+1)\,(k+2)}$$

The values of U(2), U(3) and U(4) are :

> **for k from 0 to 2 do**
U(k+2):=-1/((k+1)*(k+2))*(2*sum((l+1)*(k-l+1)*U(l+1)*U(k-l+1),l=0..k)+8*U(k));
end do;

$$\mathrm{U}(2):=\text{-}1$$

$$\mathrm{U}(3):=0$$

$$\mathrm{U}(4):=0$$

The symbolic u(x) is :

> **ux:=Sum(U(i)*x^i,i=0..4);**

$$ux:=\sum_{i=0}^{4}\mathrm{U}(i)\,x^i$$

From the induction axiom we have that U(k)=0, for k=3,4..., that means the exact solution, u(x), is :

> **ux:=sum(U(i)*x^i,i=0..4);**

$$ux := x - x^2$$

> **plot(ux, x = 0..1);**

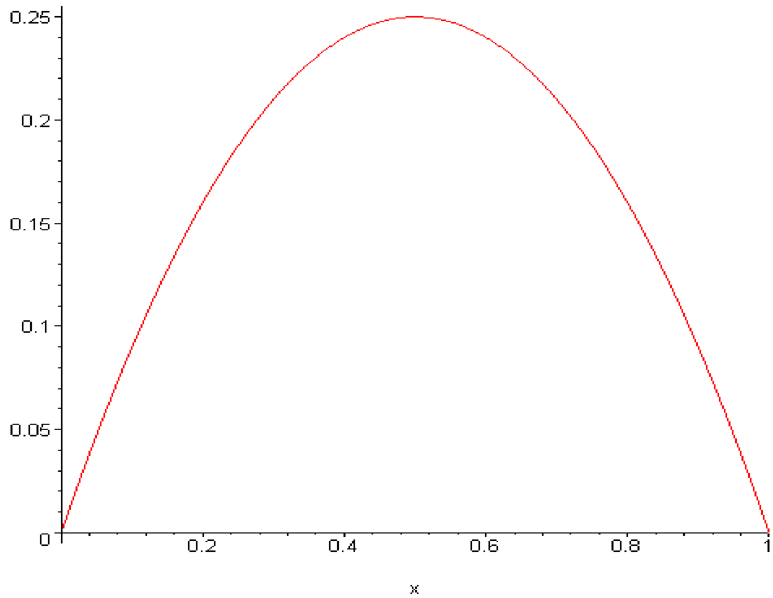

Figure: The graph of the exact solution $u(x)= x - x^2$.

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