CALCULATION OF CONVOLUTION PRODUCTS OF PIECEWISE DEFINED FUNCTIONS AND SOME APPLICATIONS

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Abstract

We present elementary proofs to some formulas given without proofs by K. A. West and J. McClellan in 1993, B. L. Evans and J. H. McClellan in 1994, and J. Cavicchi in 2002, for the calculation of the convolution integrals and sums of piecewise defined functions. Unlike "divide and conquer" strategy, these formulas are of the type "conquer what is divided".

Applications to differential equations, probability theory and linear discrete system theory are given. An example in connexion with the commutativity of the convolution product is also given. For completeness, in Annex we include a proof of the well-known elementary solution method for solving non-homogeneous linear differential equations with constant coefficients, used in the first application. The present work is part of a series of the author's articles, some of which being published in this Journal, that present the products of convolution, both in discrete and continuous case, and some of their applications.

Keywords: convolution products, piecewise defined functions, non-homogeneous linear differential equations, elementary solution method, distribution functions of random variables, linear discrete system theory.

2000 Mathematics Subject Classification: 34A30, 44A35, 60E05, 93A05.

1. Introduction

In mathematics and its applications, the functions to be used are not always as good as we would like. So, most of them are piecewise defined by different analytical formulas. Thus can be, for example, the coefficients and the right side of a non-homogeneous linear differential equation we want to solve. Of course, it can be solved separately on each interval. But this is very slow procedure. More useful is a method applicable to the matter considered as a whole. For the mentioned problem, such a method consists in computation by the formulas presented in this paper of the integral of convolution between the right side of the equation and its elementary solution. This will be made in the application given in Section 5.1.A first method for calculating convolution integrals and sums of piecewise defined functions, was given in the papers [4] and [2]. It corresponds to below corollaries. A more compact formula was given by T. J. Cavicchi in [1], its result being presented in theorem 1 from Section 2 in continuous variable case and in theorem 2 from Section 3 in discrete variable case. The main purpose of these formulas is to indicate the real limits of integration, respective summation. Because in the cited papers, the proofs are only sketched, we will give in Sections 2 the elementary but rigorous proofs of these formulas in continuous variable case. In case of discrete variable they are similar and will

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be omitted from Section 3. Both a special case and the cases when the support intervals of the factor functions have some infinite limits, are given in Section 4.

A rather difficult method for calculating the convolution integrals based on representations of the factors by step-functions, was given by I. S. Goldberg, M. G. Block and R. E. Rojas in [3].As is known, the convolution product is used in many chapters from mathematics, physics and technology. Therefore these formulas for a fast calculus of the convolutions can be used in these areas. For example, we give in Section 5 a few such applications. As noted above, in Section 5.1 we determine a particular solution of a non-homogeneous linear differential equation with piecewise constant coefficients and piecewise continuous right side. This is made by convolution between the right side of the equation and its elementary solution, both piecewise defined functions. For completeness we give in an Annex the proof of the elementary solution method.

In Section 5.2, the convolution formula is used to compute the probability distribution function of the sum of two independent random variables as convolution of their distribution functions. In Section 5.3, we present two applications to linear discrete system theory. In Section 6 is given an example which shows that although the convolution is commutative, $(f * g)(x) = (g * f)(x)$, calculation of the two products is different. In all these applications, the convolution integrals and sums between piecewise defined functions that arise, will be calculated using the formulas presented in Sections 2, 3 and 4.

Because the formulas presented in this paper are elementary, mainly consisting in calculations of integrals or sums, they can be used in the teaching process, as applications of the mathematical analysis in equations, probability and different chapters of physics.

2. Continuous Variable Case

The convolution product of two functions of real variable with complex values $f(x)$ and $g(x)$ is defined by the well-known formula

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y) dy . \qquad \text{(1c)}$$

Theorem 1c (Continuous Case). If The Functions f And g Are Integrable On The Intervals

$[l(f), r(f)]$ and $[l(g), r(g)]$ and zero otherwise, then the convolution product $f * g$ takes the form

$$(f * g)(x) = \int_{\max(l(f), x-r(g))}^{\min(r(f), x-l(g))} f(y) g(x-y) dy , \forall x \in [l(f * g), r(f * g)], \qquad \text{(2c)}$$

and zero otherwise, where $l(f * g) = l(f) + l(g)$ and $r(f * g) = r(f) + r(g)$.

Proof. When $x < l(f * g) = l(f) + l(g)$, for $y < l(f)$ we have $f(y) = 0$, and for $y \ge l(f)$ we have $x - y \le x - l(f) < l(g)$, hence $g(x - y) = 0$. Therefore, in this case we get $(f * g)(x) = 0$. Same result is obtained when $x > r(f * g)$.

Now suppose that $l(f * g) \le x \le r(f * g)$. If $y < \max(l(f), x - r(g))$, we consider the following two cases:

1) If $x < l(f) + r(g)$, then $x - r(g) < l(f)$, hence $\max(l(f), x - r(g)) = l(f)$. It results $y < l(f)$, hence $f(y) = 0$, and therefore $(f * g)(x) = 0$.

2) If $x \ge l(f) + r(g)$, then $l(f) \le x - r(g)$, hence $\max(l(f), x - r(g)) = x - r(g)$. It results $y < x - r(g)$, hence $x - y > r(g)$. We obtain $g(x - y) = 0$, hence $(f * g)(x) = 0$.

Same result is obtained when $y > \min(r(f), x - l(g))$. Therefore, formula (1c) is reduced to (2c).

We denote $\lambda(f) = r(f) - l(f)$ and $\lambda(g) = r(g) - l(g)$, the lengths of the two support intervals of the factor functions f and g, and

$$m = \min(l(f) + r(g), r(f) + l(g)),\ M = \max(l(f) + r(g), r(f) + l(g)).$$

Obviously, $l(f * g) \le m \le M \le r(f * g)$.

Corollary. *In assumptions of the theorem 1C, the convolution product is given by formulas*

$$(f * g)(x) = \int_{l(f)}^{x - l(g)} f(y) g(x - y) dy,\ \forall x \in [l(f * g), m], \quad (3c)$$

$$(f * g)(x) = \int_{x - r(g)}^{x - l(g)} f(y) g(x - y) dy,\ \forall x \in [m, M],\ \textit{if}\ \lambda(g) < \lambda(f), \quad (4c)$$

$$(f * g)(x) = \int_{l(f)}^{r(f)} f(y) g(x - y) dy,\ \forall x \in [m, M],\ \textit{if}\ \lambda(f) < \lambda(g), \quad (5c)$$

$$(f * g)(x) = \int_{x - r(g)}^{r(f)} f(y) g(x - y) dy,\ \forall x \in [M, r(f * g)]. \quad (6c)$$

Proof. If $x < m$, then $x < l(f) + r(g)$ and $x < r(f) + l(g)$, hence $x - r(g) < l(f)$ and $x - l(g) < r(f)$. In this case $\max(l(f), x - r(g)) = l(f)$ and $\min(r(f), x - l(g)) = x - l(g)$, therefore formula (2c) reduces to (3c). Analogously, if $x > M$, formula (2c) is reduced to (6c).

Now we suppose that $m \le x \le M$. If $\lambda(g) < \lambda(f)$, then $r(g) - l(g) < r(f) - l(f)$, hence

$$l(f) + r(g) = m \le x \le M = r(f) + l(g).$$

In this case $l(f) \le x - r(g)$ and $x - l(g) \le r(f)$, hence $\max(l(f), x - r(g)) = x - r(g)$ and $\min(r(f), x - l(g)) = x - l(g)$, therefore formula (2c) is reduced to (4c). Analogously, if $\lambda(f) < \lambda(g)$, formula (2c) is reduced to (5c).

Remark. Throughout the work, each function is given on its support interval, being zero otherwise.

3. Discrete Variable Case

The convolution product of two functions of integer variable with complex values $f(n)$ and $g(n)$ is defined by formula

$$(f * g)(n) = \sum_{k=-\infty}^{\infty} f(k) g(n-k). \qquad (1d)$$

As in Section 2, we can show the following discrete results:

Theorem 1D (Discrete case). *If the functions $f(n)$ and $g(n)$ are on the intervals*

$[l(f), r(f)]$ and $[l(g), r(g)]$ and zero otherwise, then the convolution product $f * g$ takes the form

$$(f * g)(n) = \sum_{k=\max(l(f), n-r(g))}^{\min(r(f), n-l(g))} f(k) g(n-k). \qquad (2d)$$

Corollary. *In assumptions of the theorem 1D, the convolution product is given by formulas*

$$(f * g)(n) = \sum_{k=l(f)}^{n-l(g)} f(k) g(n-k), \ \forall n \in [l(f * g), m], \qquad (3d)$$

$$(f * g)(n) = \sum_{k=n-r(g)}^{n-l(g)} f(k) g(n-k), \ \forall n \in [m, M], \text{ if } \lambda(g) < \lambda(f), \qquad (4d)$$

$$(f * g)(n) = \sum_{k=l(f)}^{r(f)} f(k) g(n-k), \ \forall n \in [m, M], \text{ if } \lambda(f) < \lambda(g), \qquad (5d)$$

$$(f * g)(n) = \sum_{k=n-r(g)}^{r(f)} f(k) g(n-k), \ \forall n \in [M, r(f * g)]. \qquad (6d)$$

4. Remarks

1) If $\lambda(f)=\lambda(g)$, then $m=M$ and the convolution has only cases (3) and (6).

2) Formulas given in theorem and its corollary, both for continuous and discrete variable, also apply if some of the extremities of support intervals $[l(f),r(f)]$ and $[l(g),r(g)]$ of functions f and g are infinite. In these cases, some of the formulas (3)-(6) must to be omitted. For example, if $r(f)=\infty$, then $M=r(f*g)=\infty$, hence the convolution has only the cases (3) and (4). If in addition $r(g)=\infty$, then $m=\infty$ and the convolution is calculated only with the formula (3). In this case, if $l(f)=l(g)=0$, the formula (2) is reduced to the causal convolution

$$f*g(x)=\int_0^x f(y)g(x-y)dy,\ x\geq 0,\ \text{respective}\ f*g(n)=\sum_{k=0}^{n} f(k)g(n-k),\ n\geq 0,$$

and zero otherwise.

3) A function f is named *piecewise defined* if $f=\sum_{k=1}^{n} f_k$, where f_k are functions with disjoint support intervals. If $g=\sum_{j=1}^{m} g_j$ is another such function, then $f*g=\sum_{k=1}^{n}\sum_{j=1}^{m} f_k * g_j$, hence the convolution product of such functions is reduced to the above-mentioned case.

5. Applications

5.1. Differential Equations

A particular solution for a linear non-homogeneous differential equation with constant coefficients can be obtained by the convolution between the elementary solution of the equation and its right side.

The integral convolution formulas presented in Section 2 can be used when the right side of the equation is a piecewise continuous function and its coefficients are piecewise constant functions.

Example 1. Let us determine a particular solution of the differential equation

I. $u''(x)+k(x)u(x)=v(x),\ x\in[0,\infty)$, satisfying the initial conditions $u(0)=u'(0)=0$, when

II. $k(x)=\begin{cases}1 & , & x\in\left[0,\frac{5\pi}{4}\right]\\ 4 & , & x\in\left(\frac{5\pi}{4},\infty\right)\end{cases}$, $v(x)=\begin{cases}1 & , & x\in[0,\pi]\\ x & , & x\in(\pi,\infty)\end{cases}$.

Solution. We denote $v_1(x)=1,\ x\in[0,\pi]$ and $v_2(x)=x,\ x\in(\pi,\infty)$.

If $x\in\left[0,\frac{5\pi}{4}\right]$, the equation has the form $u''(x)+u(x)=v(x)$. The elementary solution is the solution of the homogeneous equation $u''(x)+u(x)=0$, that satisfies the initial conditions $u(0)=0$, and $u'(0)=1$, hence is the function $E_1(x)=\sin x$, $\forall\, x\in[0,\infty)$.

The solution is $u(x)=v_1*E_1(x)+v_2*E_1(x)$, where

$v_1*E_1(x)=\int_{\max(0,-\infty)}^{\min(\pi,x)}v_1(y)E_1(x-y)dy$, so $v_1*E_1(x)=\int_0^x\sin(x-y)dy=1-\cos x$, $\forall\, x\in[0,\pi]$,

$v_1*E_1(x)=\int_0^\pi\sin(x-y)dy=-2\cos x$, $\forall\, x\in\left(\pi,\frac{5\pi}{4}\right]$, and

$v_2*E_1(x)=\int_{\max(\pi,-\infty)}^{\min(\infty,x)}v_2(y)E_1(x-y)dy$, so $v_2*E_1(x)=\int_\pi^x v_2(y)E_1(x-y)dy=0$, because $v_2(y)=0$, for $\forall\, x\in[0,\pi]$, $v_2*E_1(x)=\int_\pi^x y\sin(x-y)dy=x+\pi\cos x+\sin x$, $\forall\, x\in\left(\pi,\frac{5\pi}{4}\right]$.

If $x\in\left(\frac{5\pi}{4},\infty\right)$, the equation has the form $u''(x)+4u(x)=v(x)$, with the elementary solution $E_2(x)=\frac{1}{2}\sin(2x)$, $\forall\, x\in[0,\infty)$. The solution is $u(x)=v_1*E_2(x)+v_2*E_2(x)$, where

$v_1*E_2(x)=\int_{\max(0,-\infty)}^{\min(x,\pi)}v_1(y)E_2(x-y)dy=\int_0^x v_1(y)E_2(x-y)dy=0$, because $v_1(y)=0$,

$v_2*E_2(x)=\int_{\max(\pi,-\infty)}^{\min(\infty,x)}v_2(y)E_2(x-y)dy=\frac{1}{2}\int_\pi^x y\sin 2(x-y)dy=\frac{x}{4}-\frac{\pi}{4}\cos(2x)-\frac{1}{8}\sin(2x)$

.

Therefore, the solution is

$$u(x)=\begin{cases} 1-\cos x & , \quad x\in[0,\pi] \\ x+(\pi-2)\cos x+\sin x & , \quad x\in\left(\pi,\frac{5\pi}{4}\right] \\ \frac{x}{4}-\frac{\pi}{4}\cos(2x)-\frac{1}{8}\sin(2x) & , \quad x\in\left(\frac{5\pi}{4},\infty\right) \end{cases}$$

(and zero otherwise).

5.2. Probability Theory

Because the probability distribution function of the sum of two independent random variables is the convolution of their distribution functions, it can be calculated by formulas given in Section 2.

Example 2. Let X and Y be independent continuous random variables having the distributions $f(x)=3(1-x)^2$, $x\in[0,1]$ and $g(x)=1/2$, $x\in[-1,1]$. Let us determine the distribution of the sum $X+Y$.

Solution. The requested distribution is $f*g(x)=\int_{-\infty}^{\infty} f(y)g(x-y)dy$ $=\frac{3}{2}\int_{\max(0,x-1)}^{\min(1,x+1)}(1-y)^2 dy$, $\forall x\in[-1,2]$, hence $f*g(x)=\frac{3}{2}\int_0^{x+1}(1-y)^2 dy=\frac{1}{2}(x^3+1)$, $\forall x\in[-1,0]$, $f*g(x)=\frac{3}{2}\int_0^1(1-y)^2 dy=\frac{1}{2}$, $\forall x\in[0,1]$, because $L(f)=1<2=L(g)$, $f*g(x)=\frac{3}{2}\int_{x-1}^{1}(1-y)^2 dy=-\frac{1}{2}(x-2)^3$, $\forall x\in[1,2]$, the function $f*g(x)$ being zero otherwise.

5.3. System Theory

Example 3. We consider a time-invariant linear discrete system having the output $(1,0,-q^2,q^3,0,q^6,2q^7)$ when the input is $(1,-q,-q^2,2q^3)$. Determine the output corresponding to input $f(n)=q^n$, $\forall n\in[0,\alpha]$, where $q\neq 0$ is a complex number and $\alpha\neq 0$ a natural number.

Solution. The considered system is represented mathematically by a convolution operator with a sequence $h(n)$ named impulse response or weight sequence of the system. Namely, the relation between an arbitrary input $f(n)$ and its corresponding output $g(n)$ is given by the formula $g(n)=(f*h)(n)$. The transfer sequence $h(n)$ can be determined by the inverse operation of discrete convolution, named deconvolution or long division,

$$
\begin{array}{l}
1 \quad 0 \quad -q^2 \quad q^3 \quad q^4 \quad 0 \quad q^6 \quad 2q^7 \qquad\qquad 1 \quad -q \quad -q^2 \quad 2q^3 \\
\text{--------------------------------} \\
1 \quad -q \quad -q^2 \quad 2q^3 \qquad\qquad\qquad\qquad\qquad\qquad 1 \quad q \quad q^2 \quad q^3 \quad q^4 \\
\text{--------------------------------} \\
/ \quad q \quad 0 \quad -q^3 \quad q^4 \\
\quad q \quad -q^2 \quad -q^3 \quad 2q^4 \\
\text{--------------------------------} \\
\quad / \quad q^2 \quad 0 \quad -q^4 \quad 0 \\
\quad\quad q^2 \quad -q^3 \quad -q^4 \quad 2q^5 \\
\text{------------------------} \\
\quad\quad / \quad q^3 \quad 0 \quad -2q^5 \quad q^6 \\
\quad\quad\quad q^3 \quad -q^4 \quad -q^5 \quad 2q^6 \\
\text{------------------------} \\
\quad\quad\quad / \quad q^4 \quad -q^5 \quad -q^6 \quad 2q^7 \\
\quad\quad\quad\quad q^4 \quad -q^5 \quad -q^6 \quad 2q^7 \\
\text{----------------------------} \\
\quad\quad\quad\quad / \quad / \quad / \quad /
\end{array}
$$

It results $h(n)=(1,q,q^2,q^3,q^4)$. The required output is

$$g(n)=(f*h)(n)=\sum_{k=\max(0,n-4)}^{\min(\alpha,n)} q^k q^{n-k} = \sum_{k=\max(0,n-4)}^{\min(\alpha,n)} q^n = [\min(\alpha,n)-\max(0,n-4)+1]q^n =$$

$$=\begin{cases} [n+1-\max(0,n-4)]q^n & , \quad 0\le n\le\alpha, \\ [\alpha+1-\max(0,n-4)]q^n & , \quad \alpha\le n\le\alpha+4 \end{cases}$$ and zero otherwise.

For $\alpha<4$, we get $g(n)=\begin{cases} (n+1)q^n & , \quad 0\le n\le\alpha, \\ (\alpha+1)q^n & , \quad \alpha\le n\le 4, \\ (\alpha+5-n)q^n & , \quad 4\le n\le\alpha+4 \end{cases}$.

For $\alpha=4$, we get $g(n)=\begin{cases} (n+1)q^n & , \quad 0\le n\le 4, \\ (9-n)q^n & , \quad 4\le n\le 8 \end{cases}$.

For $\alpha>4$, we get $g(n)=\begin{cases} (n+1)q^n & , \quad 0\le n\le 4, \\ 5q^n & , \quad 4\le n\le\alpha, \\ (\alpha+5-n)q^n & , \quad \alpha\le n\le\alpha+4 \end{cases}$.

Example 4. *Consider a system as in example 3, having the transfer sequence* $h(n) = p^n$, $\forall n \geq \beta$. *Determine the output corresponding to input* $f(n) = q^n$, $\forall n \geq \alpha$. *Here* $p \neq 0$ *and* $q \neq 0$ *are complex numbers while* α *and* β *are integer numbers.*

Solution. The desired output is $g(n) = (f * h)(n)$, $\forall n \geq \alpha + \beta$. In this case the convolution product must be calculated only with formula (3d). For $p \neq q$ and $n \geq \alpha + \beta$, using formula for the sum of a geometric progression, we get

$$g(n) = \sum_{k=\alpha}^{n-\beta} q^k p^{n-k} = p^n \sum_{k=\alpha}^{n-\beta} \left(\frac{q}{p}\right)^k = p^n \left(\frac{q}{p}\right)^\alpha \frac{1 - \left(\frac{q}{p}\right)^{n-\beta-\alpha+1}}{1 - \frac{q}{p}} = \frac{q^\alpha p^{n+1-\alpha} - p^\beta q^{n+1-\beta}}{p-q}.$$

For $p = q$ and $n \geq \alpha + \beta$, we get

$$g(n) = \sum_{k=\alpha}^{n-\beta} p^k p^{n-k} = \sum_{k=\alpha}^{n-\beta} p^n = (n - \beta - \alpha + 1)p^n.$$

6. On The Commutativity Of The Convolution Product

As convolution product is commutative, the products $(f * g)(x)$ and $(g * f)(x)$ have the same value, but their calculation is different. We present an example that will show it in different situations.

Example 5. *Calculate the convolution products* $(f * g)(x)$ *and* $(g * f)(x)$ *for functions* $f(x) = 1$, $\forall x \in [-1,1]$ *and* $g(x) = x$, $\forall x \in [0, a]$, *with* $a > 0$, *zero otherwise.*

Solution. We have $l(f) = -1$, $r(f) = 1$, $\lambda(f) = 2$, $l(g) = 0$, $r(g) = \lambda(g) = a$, $m = \min(a-1,1)$, $M = \max(a-1,1)$, $l(f * g) = l(g * f) = -1$, $r(f * g) = r(g * f) = 1 + a$.

For $a < 2$, $m = a - 1$, $M = 1$ and $\lambda(g) < \lambda(f)$, hence from (3c), (4c) and (6c), we have $(f * g)(x) = \int_{-1}^{x} (x-y)dy = \frac{(x+1)^2}{2}$, $\forall x \in [-1, a-1]$, $(f * g)(x) = \int_{x-a}^{x} (x-y)dy = \frac{a^2}{2}$, $\forall x \in [a-1,1]$, $(f * g)(x) = \int_{x-a}^{1} (x-y)dy = \frac{a^2 - (x-1)^2}{2}$, $\forall x \in [1, 1+a]$ and from (3c), (5c) and (6c), we have $(g * f)(x) = \int_{0}^{x+1} y dy = \frac{(x+1)^2}{2}$, $\forall x \in [-1, a-1]$, $(g * f)(x) = \int_{0}^{a} y dy = \frac{a^2}{2}$, $\forall x \in [a-1,1]$, $(g * f)(x) = \int_{x-1}^{a} y dy = \frac{a^2 - (x-1)^2}{2}$, $\forall x \in [1, 1+a]$.

For $a=2$, $m=M=1$, hence from (3c) and (6c), we have $(f*g)(x)=\int_{-1}^{x}(x-y)dy=\frac{(x+1)^2}{2}, \forall x\in[-1,1]$,

$(f*g)(x)=\int_{x-2}^{1}(x-y)dy=\frac{4-(x-1)^2}{2}$, $\forall x\in[1,3]$ and from (3c) and (6c), we have $(g*f)(x)=\int_{0}^{x+1}ydy=\frac{(x+1)^2}{2}$, $\forall x\in[-1,1]$, $(g*f)(x)=\int_{x-1}^{2}ydy=\frac{4-(x-1)^2}{2}$, $\forall x\in[1,3]$.

For $a>2$, $m=1$, $M=a-1$ and $\lambda(f)<\lambda(g)$, hence from (3c), (5c) and (6c), we have $(f*g)(x)=\int_{-1}^{x}(x-y)dy=\frac{(x+1)^2}{2}, \forall x\in[-1,1], (f*g)(x)=\int_{-1}^{1}(x-y)dy=2x$, $\forall x\in[1,a-1]$, $(f*g)(x)=\int_{x-a}^{1}(x-y)dy=\frac{a^2-(x-1)^2}{2}$, $\forall x\in[1,1+a]$ and from (3c), (4c) and (6c), we have $(g*f)(x)=\int_{0}^{x+1}ydy=\frac{(x+1)^2}{2}$, $\forall x\in[-1,1]$, $(g*f)(x)=\int_{x-1}^{x+1}ydy=2x$, $\forall x\in[1,a-1]$, $(g*f)(x)=\int_{x-1}^{a}ydy=\frac{a^2-(x-1)^2}{2}$, $\forall x\in[a-1,1+a]$.

7. Annex: Solving Linear Differential Equations By Elementary Solutions

We present in the following theorem the *elementary solutions method* of solving non-homogeneous linear differential equations with constant coefficients.

Theorem. *The non-homogeneous linear differential equation*

$$\sum_{k=0}^{n} a_{n-k}u^{(k)}(x)=v(x),\ x\in[x_0,\infty) \qquad (7)$$

with constant coefficients $a_0=1$, a_k, $k=0,1,\dots,n-1$, *has a particular solution that satisfies the initial conditions* $u^{(k)}(x_0)=0$, $k=0,1,\dots,n$, *given by the convolution formula*

$$u(x)=v*E(x)=\int_{x_0}^{x}v(y)E(x-y)dy, \quad (8)$$

where $E(x)$ *is the elementary solution of equation, namely the solution of the homogeneous associated equation*

$$\sum_{k=0}^{n} a_{n-k} u^{(k)}(x) = 0\,,\ x \in [0,\infty) \quad (9)$$

that satisfies the initial conditions

$$E^{(k)}(0) = 0\,, k = 0,1,\ldots,n-2\,, \qquad (10)$$

$$E^{(n-1)}(0) = 1\,. \quad (11)$$

Proof. If $u(x)$ is given by formula (8), by differentiation of the integral with respect to its parameter x and taking into account the initial conditions (10) and (11), we obtain

$$u^{(k)}(x) = \int_{x_0}^{x} v(y) E^{(k)}(x-y)dy\,,\ \forall\, k = 1,\ldots,n-1, \quad (12)$$

$$u^{(n)}(x) = \int_{x_0}^{x} v(y) E^{(n)}(x-y)dy + v(x)\,. \qquad (13)$$

Using relations (8), (12), (13), it results

$$\sum_{k=0}^{n} a_{n-k} u^{(k)}(x) = \sum_{k=0}^{n} a_{n-k} \int_{x_0}^{x} v(y) E^{(k)}(x-y)dy + v(x) =$$

$$= \int_{x_0}^{x} v(y) \sum_{k=0}^{n} a_{n-k} E^{(k)}(x-y)dy + v(x) = v(x)\,,$$

the last equality resulting from the relation (9) applied for $u(x) = E(x-y)$. Thus, the function $u(x)$ given by formula (8) is solution of equation (7). From (8) and (12), it also results that $u(x)$ has zero initial conditions.

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